KUSOR-3D — Validation of the static solver

357 of 357 values within tolerance, in 43 cases. Each case has one page: what it proves, the model with its supports and its load, the data, the reference solution, the results and a conclusion.

In parts 1 to 3 the references are closed-form solutions of the Timoshenko beam, and a value passes when it differs from its reference by less than 1e-06 in relative terms. In part 4 the references are published values, met within 5e-04 (1e-02 where the published value is printed with two figures).

Supports are written with the six displacements of a node: Ux, Uy, Uz (translations) and Rx, Ry, Rz (rotations). Units: N, m, Pa, rad.

What each part covers

Part 1 — The beam element. KUSOR uses a Timoshenko beam, which deforms in bending and in shear. Each case isolates one stiffness of that beam on a simple section, a rectangle or a solid circle. The beams are short on purpose (L/a = 5), so that shear deformation is visible in the results.

Part 2 — Loads on a member. In KUSOR a load is applied directly on a member: uniform, over part of its length, varying linearly, or concentrated at a point, in the axes of the model or in those of the member. Each case checks the displacements, the reactions and the forces inside the loaded member, with shear deformation.

Part 3 — Supports and connections. Supports that are elastic in translation or in rotation, hinges made by releasing a member at one of its ends, and member offsets, which are rigid arms between a node and the end of a member. Each case has a closed-form solution in which the spring, the hinge or the offset changes the result visibly.

Part 4 — Published cases. Structures whose results are published. Cases 4.1 to 4.5 come from the validation manual of code_aster (fascicule V3.01), most of them taken from the AFNOR guide for the validation of structural analysis programs; cases 4.6 to 4.17 come from the exercise manual of RDM – Ossatures (Y. Debard, IUT du Mans), with results from textbooks and closed-form solutions. Where a source is incomplete or contradicts itself, the choice made is written on the page.

Contents

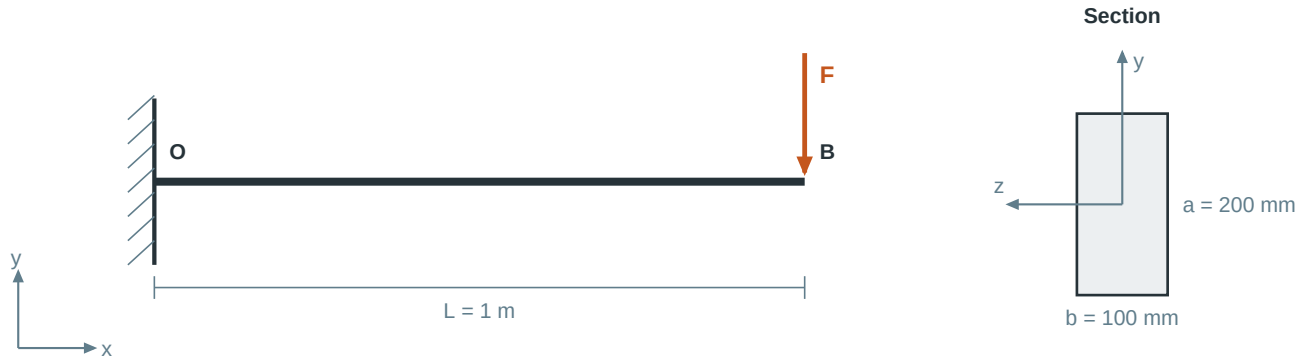
Case	Subject	Values checked	Result	Page
Part 1	**The beam element**			
1.1	Cantilever, transverse force at the tip (strong axis)	5	OK	3
1.2	Cantilever, transverse force at the tip (weak axis)	5	OK	4
1.3	Cantilever, axial force at the tip	3	OK	5
1.4	Cantilever, torque at the tip	2	OK	6
1.5	Cantilever, moment at the tip (strong axis)	5	OK	7
1.6	Cantilever, moment at the tip (weak axis)	5	OK	8
1.7	Beam on two simple supports, force at mid-span	6	OK	9
1.8	Beam clamped at both ends, force at mid-span	6	OK	10
1.9	Cantilever of circular section, transverse force at the tip	5	OK	11
1.10	Cantilever that follows no axis of the model	7	OK	12
Part 2	**Loads on a member**			
2.1	Cantilever, uniform load over the whole span	9	OK	13
2.2	Beam clamped at both ends, uniform load	9	OK	14
2.3	Beam on two simple supports, uniform load	8	OK	15
2.4	Cantilever, uniform load over part of the length	8	OK	16
2.5	Beam on two simple supports, load varying linearly	8	OK	17
2.6	Cantilever, force applied between the nodes	9	OK	18
2.7	Beam clamped at both ends, force applied between the nodes	7	OK	19
2.8	Inclined member, load given in the axes of the member	9	OK	20
2.9	Inclined member, load given in the axes of the model	9	OK	21

Case	Subject	Values checked	Result	Page
Part 3	**Supports and connections**			
3.1	Cantilever with its tip on a spring	5	**OK**	22
3.2	Cantilever on a support that is elastic in rotation	6	**OK**	23
3.3	Beam clamped at both ends with a hinge at mid-span	9	**OK**	24
3.4	Released member under a uniform load	8	**OK**	25
3.5	Member offset along its axis, at the clamp	7	**OK**	26
3.6	Member offset across its axis, at the loaded end	7	**OK**	27
3.7	Member offset to one side of its nodes, under an axial force	8	**OK**	28
Part 4	**Published cases**			
4.1	SSLL01 — Slender beam clamped at both ends	7	**OK**	29
4.2	SSLL04 — Space structure on elastic supports	9	**OK**	30
4.3	SSLL10 — Frame with lateral links	5	**OK**	31
4.4	SSLL14 — Portal frame pinned at its feet	20	**OK**	32
4.5	SSLL102 — Clamped beam under unit loads	20	**OK**	33
4.6	RDM, example 1 — Plane portal frame	12	**OK**	34
4.7	RDM, example 3 — Plane ring	7	**OK**	35
4.8	RDM, example 4 — Grid of beams	18	**OK**	36
4.9	RDM, example 5 — Space frame	21	**OK**	37
4.10	RDM, example 7 — Pin-ended members loaded between their nodes	12	**OK**	38
4.11	RDM, exercise E1 — Plane truss	11	**OK**	39
4.12	RDM, exercise E2 — Portal with loads on its three members	4	**OK**	40
4.13	RDM, exercise E3 — Portal with three hinges	5	**OK**	41
4.14	RDM, exercise E4 — Three-hinged portal with a sloping roof	4	**OK**	42
4.15	RDM, exercise E8 — Three bars, a load across one of them	5	**OK**	43
4.16	RDM, exercise E10 — Space truss with loaded members	18	**OK**	44
4.17	RDM, exercise S11 — Bending, torsion and shear together	4	**OK**	45

Case 1.1 — Cantilever, transverse force at the tip (strong axis)

What this case proves. The bending and shear stiffness of the beam about its strong axis: the deflection of a short cantilever is the sum of a bending part and a shear part.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F	10000 N at B, along –y
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$$u_y(B) = -F L^3 / (3 E I_z) - F L / (G A_s) \quad \text{(bending, then shear)}$$

$$\theta_z(B) = -F L^2 / (2 E I_z)$$

$$R_y(O) = F \qquad M_z(O) = F L$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_y [m]	$-F L^3/(3 E I_z) - F L/(G A_s)$	-0.0002578	-0.0002578	4.2e-16	OK
Rotation at B, θ_z [rad]	$-F L^2/(2 E I_z)$	-0.000375	-0.000375	4.3e-16	OK
Reaction at O, R_y [N]	F	10000	10000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	F L	10000	10000	0.0e+00	OK
Deflection at B without shear areas [m]	$-F L^3/(3 E I_z)$	-0.00025	-0.00025	0.0e+00	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

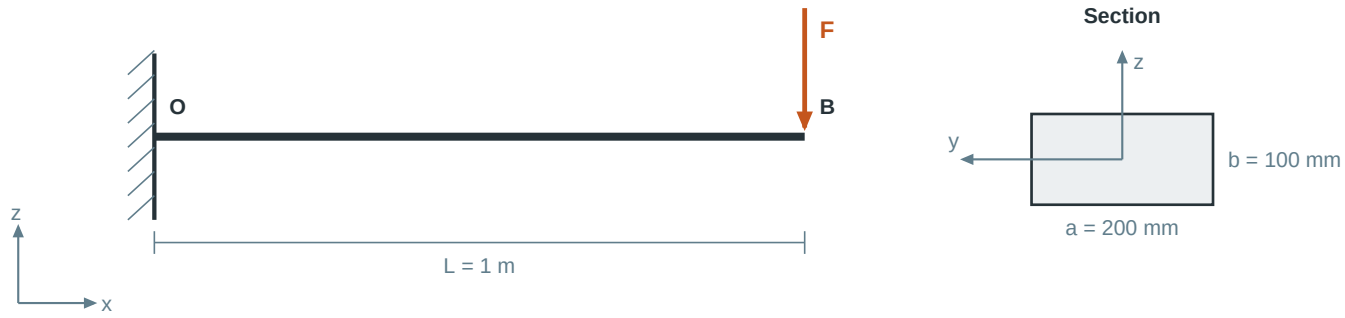
Conclusion

The bending term is 0.00025 m and the shear term 7.8e-06 m: shear adds 3.1 % to the deflection of this short beam (L/a = 5), far more than the tolerance, so a beam without shear deformation would fail this case. KUSOR reproduces the Timoshenko value, and gives the Euler-Bernoulli value when the shear areas are zero.

Case 1.2 — Cantilever, transverse force at the tip (weak axis)

What this case proves. The same as case 1.1 about the other axis of the section: the force is along z, the beam bends about y with the smaller second moment of area.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	$I_y = a\,b^3/12$	1.6667e-05 m^4
Shear area	$A_s = 5/6\ A$	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F	10000 N at B, along –z
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$$u_z(B) = -F L^3 / (3 E I_y) - F L / (G A_s)$$

$\theta_y(B) = F L^2 / (2 E I_y)$ (a deflection along –z turns the section positively about y)

$R_z(O) = F \qquad M_y(O) = -F L$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_z [m]	$-F L^3/(3 E I_y) - F L/(G A_s)$	-0.0010078	-0.0010078	2.2e-16	OK
Rotation at B, θ_y [rad]	$F L^2/(2 E I_y)$	0.0015	0.0015	0.0e+00	OK
Reaction at O, R_z [N]	F	10000	10000	0.0e+00	OK
Clamping moment at O, M_y [N.m]	–F L	-10000	-10000	0.0e+00	OK
Deflection at B without shear areas [m]	$-F L^3/(3 E I_y)$	-0.001	-0.001	0.0e+00	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

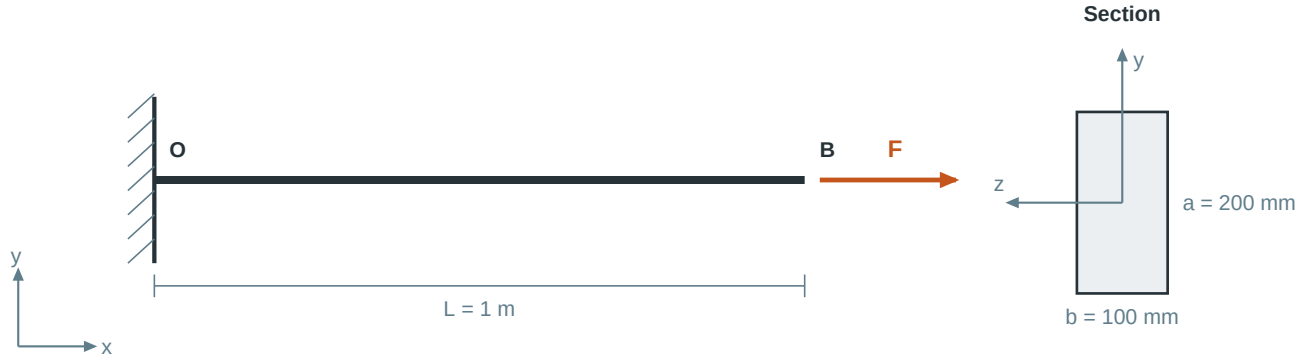
Conclusion

The deflection is 0.001008 m, four times that of case 1.1 for the bending part, as I_y is a quarter of I_z; the shear part is the same (7.8e-06 m). The two bending axes of the beam are not mixed up, and the signs of the rotation and of the clamping moment follow the right-hand rule about y.

Case 1.3 — Cantilever, axial force at the tip

What this case proves. The axial stiffness E A / L of the beam, and that an axial force alone produces no deflection.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F	10000 N at B, along +x
Mesh		1 element, 2 nodes

Reference solution

$$u_x(B) = F\,L / (E\,A)$$

$$R_x(O) = -F$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Elongation at B, u_x [m]	F L/(E A)	2.5e-06	2.5e-06	0.0e+00	OK
Reaction at O, R_x [N]	–F	-10000	-10000	0.0e+00	OK
Deflection at B, u_y [m]	0	0	0	0.0e+00	OK

Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

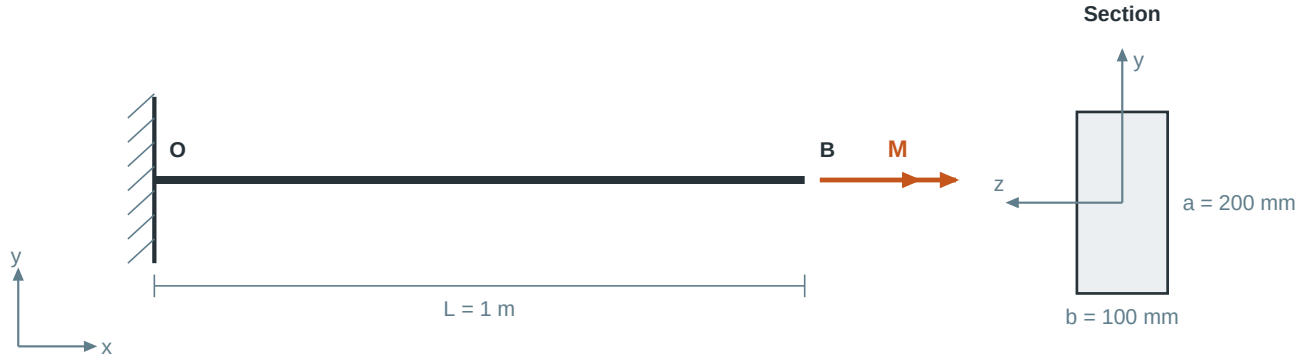
Conclusion

The elongation is F L / (E A) and the beam stays straight: tension and bending are uncoupled for a section loaded at its centroid.

Case 1.4 — Cantilever, torque at the tip

What this case proves. The torsional stiffness G J / L of the beam (uniform, Saint-Venant torsion).

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Torsion constant	J = 0.229 a b^3	4.58e-05 m^4
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	M	10000 N.m at B, about +x
Mesh		1 element, 2 nodes

Reference solution

$$\theta_x(B) = M\,L\,/\,(G\,J)$$

$$M_x(O) = -\,M$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Twist at B, θ_x [rad]	M L/(G J)	0.002838428	0.002838428	0.0e+00	OK
Reaction at O, M_x [N.m]	–M	-10000	-10000	0.0e+00	OK

Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

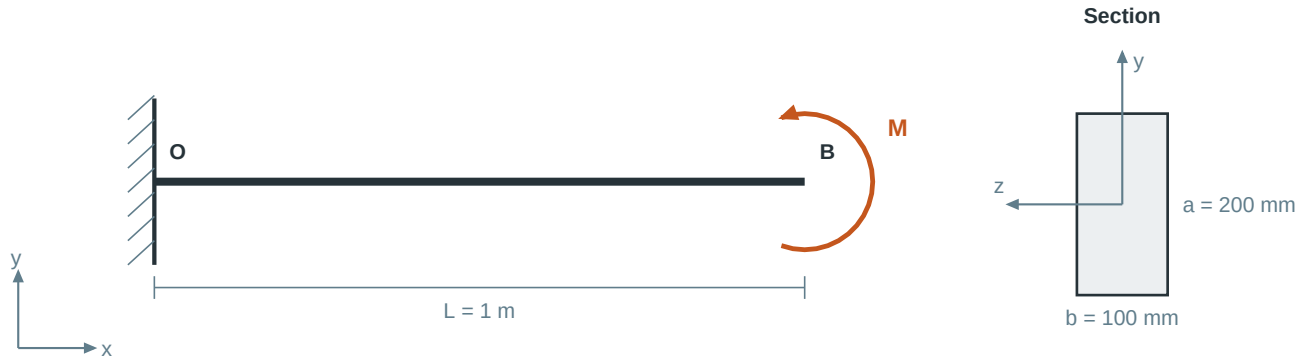
Conclusion

The twist is M L / (G J). The torsion constant is a datum here (J = 0.229 a b^3 for a rectangle of sides 2 to 1): this case checks the beam, the calculation of J belongs to the validation of the sections. Warping restraint is not part of the element.

Case 1.5 — Cantilever, moment at the tip (strong axis)

What this case proves. Pure bending about the strong axis: the bending moment is constant, there is no shear force, so shear deformation must play no part.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	$I_z = b\ a^3/12$	6.6667e-05 m^4
Shear area	$A_s = 5/6\ A$	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	M	10000 N.m at B, about +z
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$$u_y(B) = M\ L^2 / (2\ E\ I_z)$$

$$\theta_z(B) = M\ L / (E\ I_z)$$

$$M_z(O) = -M \qquad R_y(O) = 0$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_y [m]	$M\ L^2/(2\ E\ I_z)$	0.000375	0.000375	5.8e-16	OK
Rotation at B, θ_z [rad]	$M\ L/(E\ I_z)$	0.00075	0.00075	5.8e-16	OK
Clamping moment at O, M_z [N.m]	–M	-10000	-10000	0.0e+00	OK
Reaction at O, R_y [N]	0	0	0	0.0e+00	OK
Deflection at B without shear areas [m]	$M\ L^2/(2\ E\ I_z)$	0.000375	0.000375	0.0e+00	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

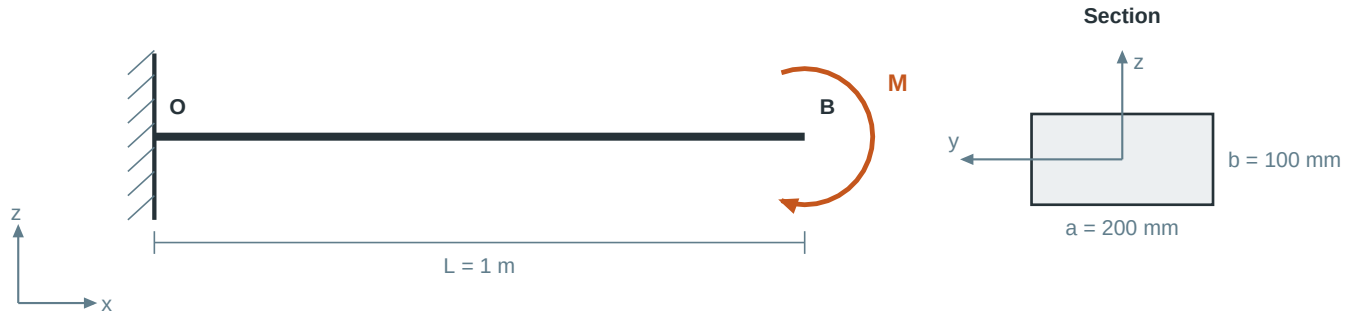
Conclusion

With and without shear areas the deflection is the same, $M\ L^2 / (2\ E\ I_z)$: the shear term of the element appears only where there is a shear force.

Case 1.6 — Cantilever, moment at the tip (weak axis)

What this case proves. Pure bending about the weak axis, and the sign that links a moment about y to a deflection along z.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	$I_y = a\ b^3/12$	1.6667e-05 m^4
Shear area	$A_s = 5/6\ A$	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	M	10000 N.m at B, about +y
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$u_z(B) = -\,M\,L^2 / (2\,E\,I_y)$ (a positive moment about y lowers the tip)

$\theta_y(B) = M\,L / (E\,I_y)$

$M_y(O) = -\,M \qquad R_z(O) = 0$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_z [m]	$-M\,L^2/(2\,E\,I_y)$	-0.0015	-0.0015	0.0e+00	OK
Rotation at B, θ_y [rad]	$M\,L/(E\,I_y)$	0.003	0.003	0.0e+00	OK
Clamping moment at O, M_y [N.m]	−M	-10000	-10000	0.0e+00	OK
Reaction at O, R_z [N]	0	0	0	0.0e+00	OK
Deflection at B without shear areas [m]	$-M\,L^2/(2\,E\,I_y)$	-0.0015	-0.0015	0.0e+00	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

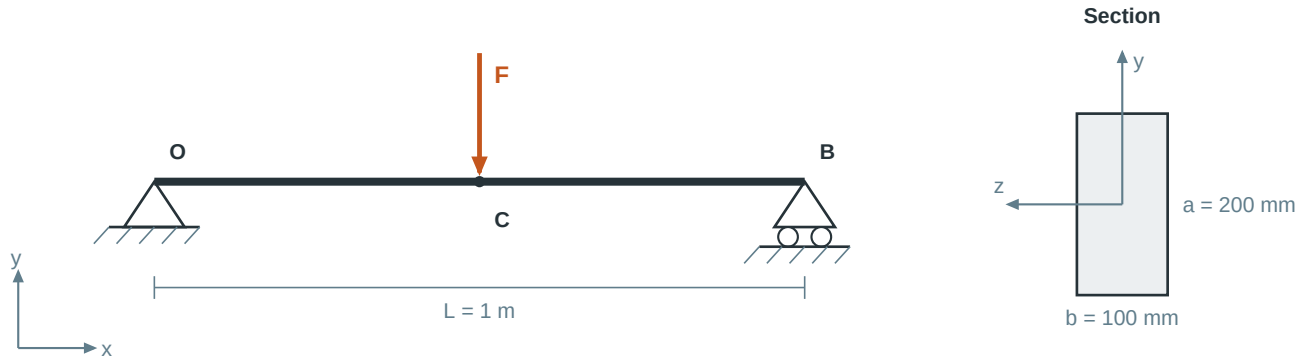
Conclusion

The deflection is along −z for a positive moment about y, as the right-hand rule requires, and it does not depend on the shear areas.

Case 1.7 — Beam on two simple supports, force at mid-span

What this case proves. Two elements joined at a loaded node, on supports that leave the rotations free: the shear part of the deflection is now a larger share of the total.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = 0 B: Uy = Uz = 0
Load	F	10000 N at C (mid-span), along –y
Mesh		2 elements, 3 nodes (O, C at mid-span, B)

Reference solution (Timoshenko beam)

$$u_y(C) = -F L^3 / (48 E I_z) - F L / (4 G A_s)$$

$$\theta_z(O) = -F L^2 / (16 E I_z) = -\theta_z(B)$$

$$R_y(O) = R_y(B) = F / 2$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at C, u_y [m]	$-F L^3/(48 E I_z) - F L/(4 G A_s)$	-1.7575e-05	-1.7575e-05	0.0e+00	OK
Rotation at O, θ_z [rad]	$-F L^2/(16 E I_z)$	-4.6875e-05	-4.6875e-05	0.0e+00	OK
Rotation at B, θ_z [rad]	$F L^2/(16 E I_z)$	4.6875e-05	4.6875e-05	0.0e+00	OK
Reaction at O, R_y [N]	F/2	5000	5000	0.0e+00	OK
Reaction at B, R_y [N]	F/2	5000	5000	0.0e+00	OK
Deflection at C without shear areas [m]	$-F L^3/(48 E I_z)$	-1.5625e-05	-1.5625e-05	2.2e-16	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

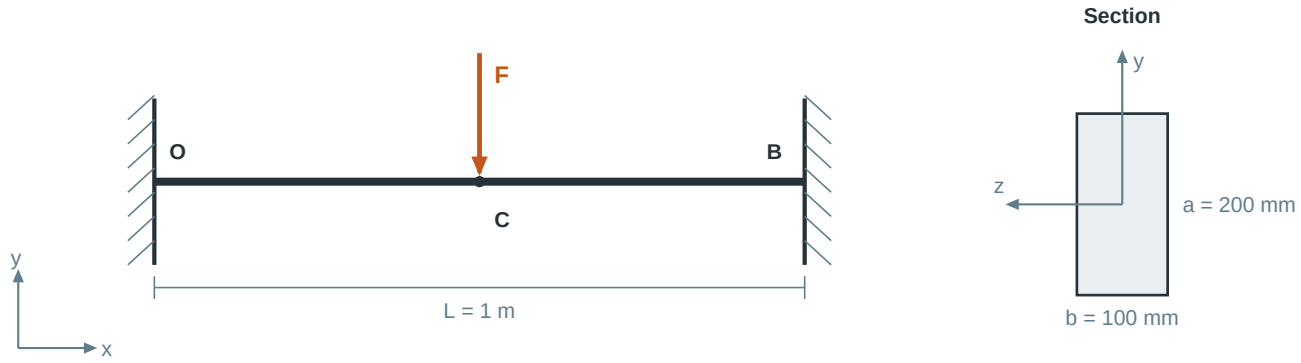
Conclusion

Shear adds 12.5 % to the deflection (bending 1.562e-05 m, shear 1.95e-06 m). The rotations at the supports are those of the section and do not depend on shear; the reactions are F/2 each. Rx is held at O only, which is enough to prevent the beam from spinning about its axis.

Case 1.8 — Beam clamped at both ends, force at mid-span

What this case proves. A statically indeterminate beam: the clamping moments come from the stiffness of the element, and the shear part of the deflection is half of the bending part.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	$I_z = b\ a^3/12$	6.6667e-05 m^4
Shear area	$A_s = 5/6\ A$	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O and B: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F	10000 N at C (mid-span), along –y
Mesh		2 elements, 3 nodes (O, C at mid-span, B)

Reference solution (Timoshenko beam)

$$u_y(C) = -F L^3 / (192 E I_z) - F L / (4 G A_s)$$

$$M_z(O) = F L / 8 = -M_z(B) \qquad R_y(O) = R_y(B) = F / 2$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at C, u_y [m]	$-F L^3/(192 E I_z) - F L/(4 G A_s)$	-5.85625e-06	-5.85625e-06	5.8e-16	OK
Rotation at C, θ_z [rad]	0	0	0	0.0e+00	OK
Reaction at O, R_y [N]	F/2	5000	5000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	F L/8	1250	1250	0.0e+00	OK
Clamping moment at B, M_z [N.m]	–F L/8	-1250	-1250	0.0e+00	OK
Deflection at C without shear areas [m]	$-F L^3/(192 E I_z)$	-3.90625e-06	-3.90625e-06	4.3e-16	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

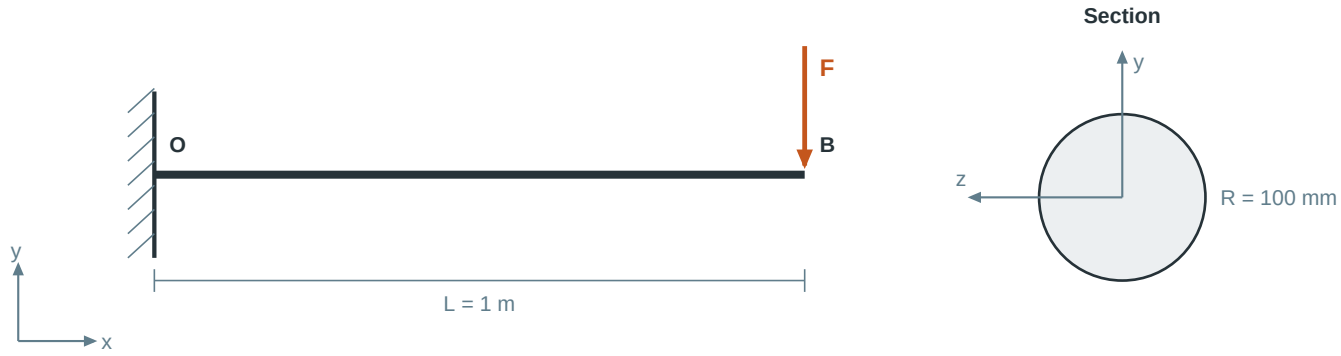
Conclusion

Shear adds 50 % to the deflection of this short clamped beam (bending 3.906e-06 m, shear 1.95e-06 m): it is the case of this part where an element without shear deformation would be most wrong. By symmetry the clamping moments stay F L / 8, with or without shear.

Case 1.9 — Cantilever of circular section, transverse force at the tip

What this case proves. Case 1.1 on the second simple section, with its own shear factor: nothing in the element is tied to the rectangle.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	solid circle	R = 100 mm
Area	$A = \pi R^2$	0.031416 m^2
Second moment of area	$I = \pi R^4/4$	7.854e-05 m^4
Shear area	$A_s = 0.9\ A$	0.028274 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F	10000 N at B, along –y
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$$u_y(B) = -F L^3 / (3 E I) - F L / (G A_s)$$

$$\theta_z(B) = -F L^2 / (2 E I)$$

$$R_y(O) = F \qquad M_z(O) = F L$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_y [m]	$-F L^3/(3 E I) - F L/(G A_s)$	-0.0002168044	-0.0002168044	0.0e+00	OK
Rotation at B, θ_z [rad]	$-F L^2/(2 E I)$	-0.0003183099	-0.0003183099	3.4e-16	OK
Reaction at O, R_y [N]	F	10000	10000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	F L	10000	10000	0.0e+00	OK
Deflection at B without shear areas [m]	$-F L^3/(3 E I)$	-0.0002122066	-0.0002122066	2.6e-16	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

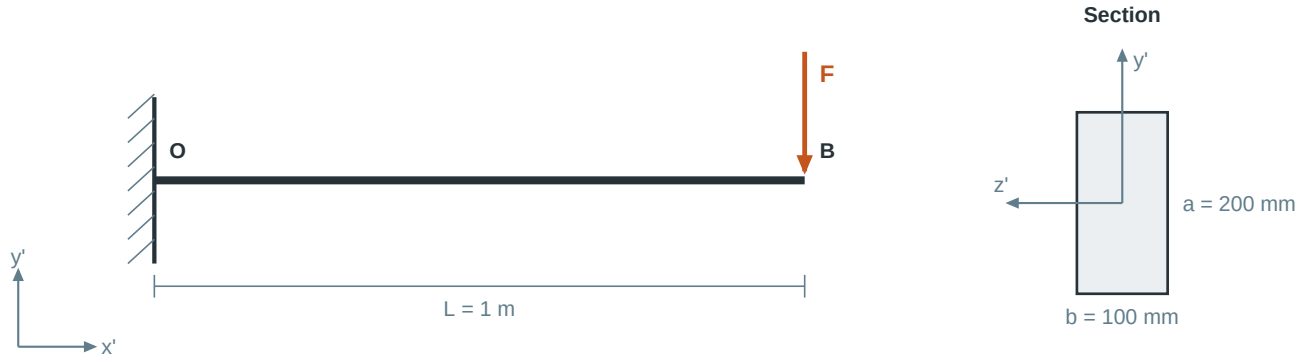
Conclusion

Bending 0.0002122 m, shear 4.598e-06 m (2.2 % more). The shear factor 0.9 is that of the code_aster case SSLL102 for a solid circle; it is a datum of the model.

Case 1.10 — Cantilever that follows no axis of the model

What this case proves. Case 1.1 again, the member now pointing along (1, 1, 1): the result in the axes of the member must be unchanged, and its components in the axes of the model must be those of the rotation between the two.

Model



Data	Symbol	Value
Length	L	1.000 m
Direction of the member	x'	(1, 1, 1)/√3 in the axes of the model
Axes of the section	y', z'	y' = (–1, 1, 0)/√2 z' = (–1, –1, 2)/√6
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	$I_z = b\ a^3/12$	6.6667e-05 m^4
Shear area	$A_s = 5/6\ A$	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F	10000 N at B, along –y'
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$$u_{y'}(B) = -F L^3 / (3 E I_z) - F L / (G A_s) \quad \text{(as in case 1.1)}$$

$$U(B) = u_{y'}\, y' \quad \text{so} \quad U_x = -u_{y'}/\sqrt{2},\ \ U_y = u_{y'}/\sqrt{2},\ \ U_z = 0$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B along y' [m]	$-F L^3/(3 E I_z) - F L/(G A_s)$	-0.0002578	-0.0002578	3.2e-15	OK
Rotation at B about z' [rad]	$-F L^2/(2 E I_z)$	-0.000375	-0.000375	3.5e-15	OK
Deflection at B along z' [m]	0	0	2.30523e-18	0.0e+00	OK
Displacement at B, U_x [m]	$-u_{y'}/\sqrt{2}$	0.0001822921	0.0001822921	8.0e-15	OK
Displacement at B, U_y [m]	$u_{y'}/\sqrt{2}$	-0.0001822921	-0.0001822921	2.2e-15	OK
Displacement at B, U_z [m]	0	0	1.882078e-18	0.0e+00	OK
Deflection at B along y' without shear areas [m]	$-F L^3/(3 E I_z)$	-0.00025	-0.00025	2.6e-15	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

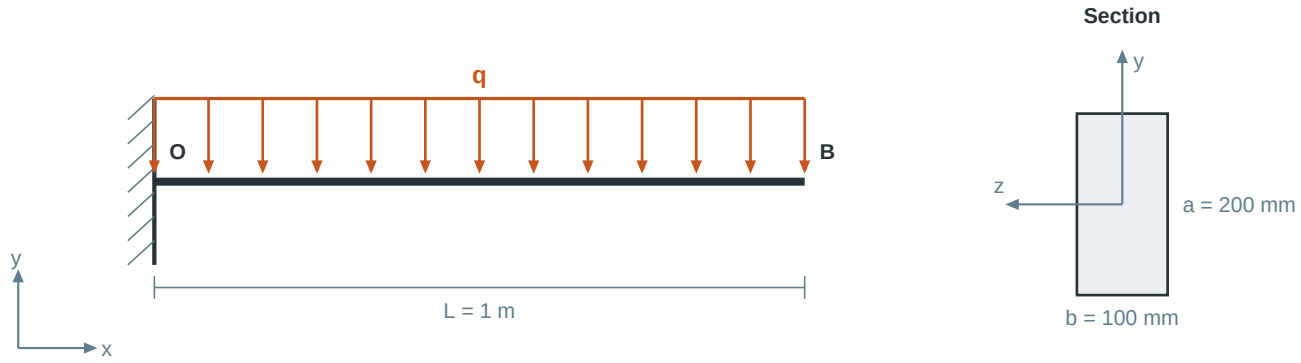
Conclusion

In the axes of the member the deflection is that of case 1.1 to the last figure, and nothing appears along z'. The transformation between the axes of the member and those of the model is therefore right for a member in a general position.

Case 2.1 — Cantilever, uniform load over the whole span

What this case proves. A uniform load applied on a member, modelled with a single element: the displacements, the reactions and the forces inside the member are those of the loaded beam.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	q	10000 N/m over the whole member, along –y
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$u_y(B) = – q L^4 / (8 E I_z) – q L^2 / (2 G A_s)$ (bending, then shear)

$\theta_z(B) = – q L^3 / (6 E I_z)$ $R_y(O) = q L$ $M_z(O) = q L^2 / 2$

In the member: $V_y(O) = q L$, $M_z(O) = – q L^2 / 2$ (the top fibre is in tension), $V_y(B) = M_z(B) = 0$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_y [m]	$–q L^4/(8 E I_z) – q L^2/(2 G A_s)$	-9.765e-05	-9.765e-05	1.4e-16	OK
Rotation at B, θ_z [rad]	$–q L^3/(6 E I_z)$	-0.000125	-0.000125	0.0e+00	OK
Reaction at O, R_y [N]	q L	10000	10000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	$q L^2/2$	5000	5000	0.0e+00	OK
Member at O, shear V_y [N]	q L	10000	10000	0.0e+00	OK
Member at O, moment M_z [N.m]	$–q L^2/2$	-5000	-5000	0.0e+00	OK
Member at B, shear V_y [N]	0	0	0	0.0e+00	OK
Member at B, moment M_z [N.m]	0	0	0	0.0e+00	OK
Deflection at B without shear areas [m]	$–q L^4/(8 E I_z)$	-9.375e-05	-9.375e-05	1.4e-16	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

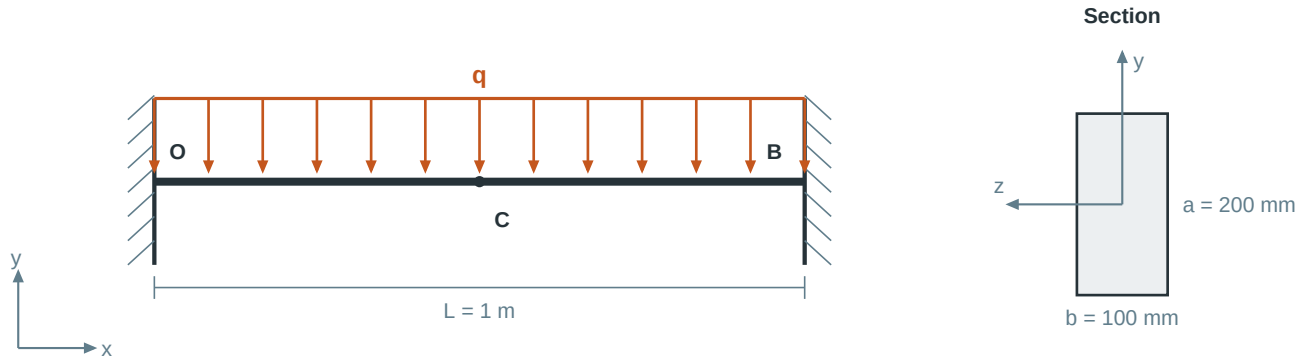
Conclusion

One element is enough to carry a distributed load: the member need not be cut into pieces. The displacements are exact, shear included (it adds 4.2 % here), and the forces inside the member are q L and $q L^2/2$ at the clamp and zero at the free end. The bending moment of a member is negative when the fibre towards +y is in tension.

Case 2.2 — Beam clamped at both ends, uniform load

What this case proves. A uniform load on a statically indeterminate beam: the clamping moments q L^2/12 and the moment q L^2/24 at mid-span, with the shear part of the deflection.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O and B: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	q	10000 N/m over both elements, along –y
Mesh		2 elements, 3 nodes (O, C at mid-span, B)

Reference solution (Timoshenko beam)

$$u_y(C) = -\,q\,L^4 / (384\,E\,I_z) - q\,L^2 / (8\,G\,A_s)$$

$$M_z(O) = q\,L^2 / 12 = -\,M_z(B) \qquad R_y(O) = R_y(B) = q\,L / 2$$

In the member: $M_z(O) = -\,q\,L^2 / 12$, $M_z(C) = q\,L^2 / 24$, $V_y(C) = 0$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at C, u_y [m]	$-q\,L^4/(384\,E\,I_z) - q\,L^2/(8\,G\,A_s)$	-2.928125e-06	-2.928125e-06	5.8e-16	OK
Reaction at O, R_y [N]	q L/2	5000	5000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	q L^2/12	833.3333	833.333	4.0e-07	OK
Clamping moment at B, M_z [N.m]	–q L^2/12	-833.3333	-833.333	4.0e-07	OK
Member at O, shear V_y [N]	q L/2	5000	5000	0.0e+00	OK
Member at O, moment M_z [N.m]	–q L^2/12	-833.3333	-833.333	4.0e-07	OK
Member at C, moment M_z [N.m]	q L^2/24	416.6667	416.667	8.0e-07	OK
Member at C, shear V_y [N]	0	0	0	0.0e+00	OK
Deflection at C without shear areas [m]	$-q\,L^4/(384\,E\,I_z)$	-1.953125e-06	-1.953125e-06	4.3e-16	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

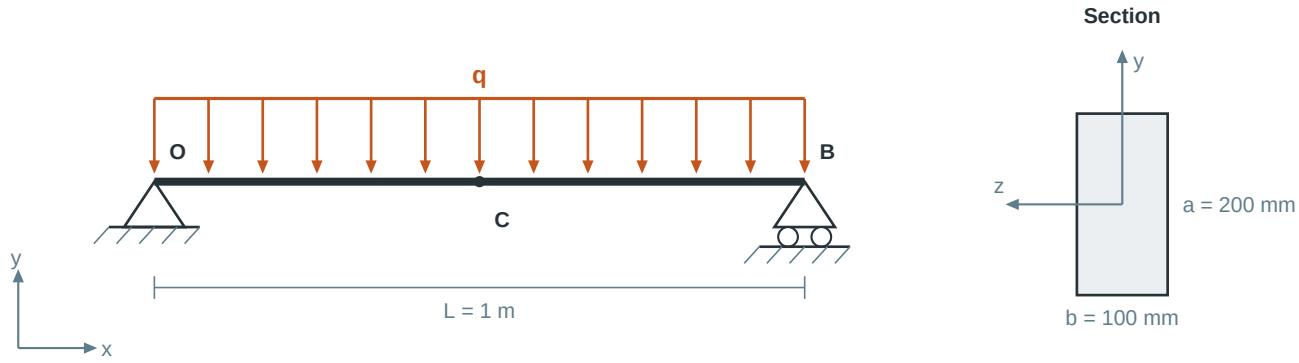
Conclusion

Shear adds 50 % to the deflection at mid-span (bending 1.953e-06 m, shear 9.75e-07 m). The moment in the member changes sign between the clamp (– q L^2/12, top fibre in tension) and mid-span (+ q L^2/24), and the shear force vanishes at mid-span, as symmetry requires.

Case 2.3 — Beam on two simple supports, uniform load

What this case proves. The most common loaded beam: the deflection 5 q L^4/384 E I with its shear part, the rotations at the supports and the moment q L^2/8 at mid-span.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = 0 B: Uy = Uz = 0
Load	q	10000 N/m over both elements, along –y
Mesh		2 elements, 3 nodes (O, C at mid-span, B)

Reference solution (Timoshenko beam)

$$u_y(C) = -\,5\,q\,L^4 / (384\,E\,I_z) - q\,L^2 / (8\,G\,A_s)$$

$$\theta_z(O) = -\,q\,L^3 / (24\,E\,I_z) = -\,\theta_z(B) \qquad R_y(O) = R_y(B) = q\,L / 2$$

In the member: $M_z(C) = q\,L^2 / 8$ (the bottom fibre is in tension), $V_y(C) = 0$, $V_y(B) = -\,q\,L / 2$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at C, u_y [m]	$-5\,q\,L^4/(384\,E\,I_z) - q\,L^2/(8\,G\,A_s)$	-1.074062e-05	-1.074062e-05	1.6e-16	OK
Rotation at O, θ_z [rad]	$-q\,L^3/(24\,E\,I_z)$	-3.125e-05	-3.125e-05	0.0e+00	OK
Rotation at B, θ_z [rad]	$q\,L^3/(24\,E\,I_z)$	3.125e-05	3.125e-05	0.0e+00	OK
Reaction at O, R_y [N]	q L/2	5000	5000	0.0e+00	OK
Member at C, moment M_z [N.m]	$q\,L^2/8$	1250	1250	0.0e+00	OK
Member at C, shear V_y [N]	0	0	0	0.0e+00	OK
Member at B, shear V_y [N]	–q L/2	-5000	-5000	0.0e+00	OK
Deflection at C without shear areas [m]	$-5\,q\,L^4/(384\,E\,I_z)$	-9.765625e-06	-9.765625e-06	0.0e+00	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

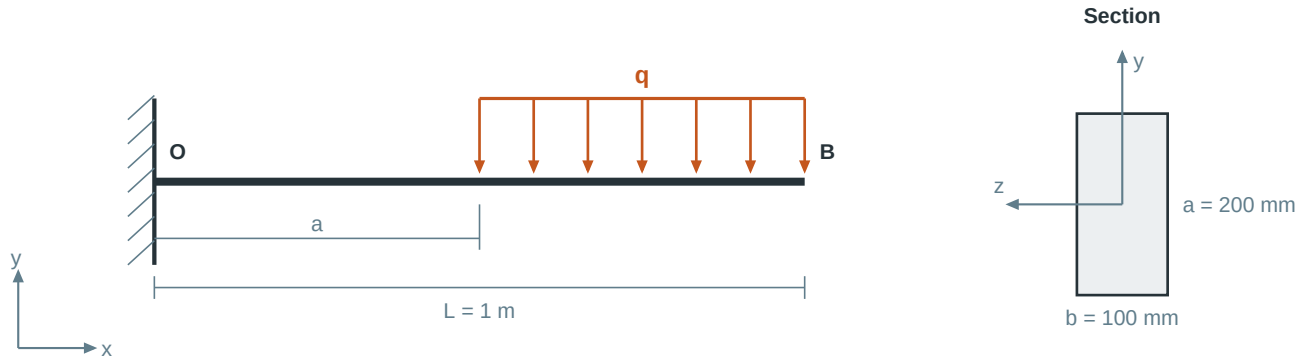
Conclusion

Shear adds 10.0 % to the deflection (bending 9.766e-06 m, shear 9.75e-07 m). The rotations at the supports do not depend on shear. The shear force goes from + q L/2 at O to – q L/2 at B and the moment at mid-span is positive: the fibre towards –y is in tension.

Case 2.4 — Cantilever, uniform load over part of the length

What this case proves. A load that covers only part of a member, here its outer half, on a single element: no node is needed where the load starts.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	$I_z = b\ a^3/12$	6.6667e-05 m^4
Shear area	$A_s = 5/6\ A$	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	q, a	10000 N/m along –y, from a = 0.500 m to the end B
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$$u_y(B) = -\,q\,(3\,L^4 - 4\,L\,a^3 + a^4)\,/\,(24\,E\,I_z) - q\,(L-a)(L+a)\,/\,(2\,G\,A_s)$$

$$\theta_z(B) = -\,q\,(L^3 - a^3)\,/\,(6\,E\,I_z)$$

$$R_y(O) = q\,(L-a) \qquad M_z(O) = q\,(L-a)(L+a)\,/\,2 \quad \text{(the resultant acts at } (L+a)/2\text{)}$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_y [m]	see the reference solution	-8.300312e-05	-8.300312e-05	4.9e-16	OK
Rotation at B, θ_z [rad]	$-q\,(L^3 - a^3)/(6\,E\,I_z)$	-0.000109375	-0.000109375	3.7e-16	OK
Reaction at O, R_y [N]	$q\,(L-a)$	5000	5000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	$q\,(L-a)(L+a)/2$	3750	3750	0.0e+00	OK
Member at O, shear V_y [N]	$q\,(L-a)$	5000	5000	0.0e+00	OK
Member at O, moment M_z [N.m]	$-q\,(L-a)(L+a)/2$	-3750	-3750	0.0e+00	OK
Member at B, moment M_z [N.m]	0	0	0	0.0e+00	OK
Deflection at B without shear areas [m]	bending term alone	-8.007812e-05	-8.007812e-05	3.4e-16	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

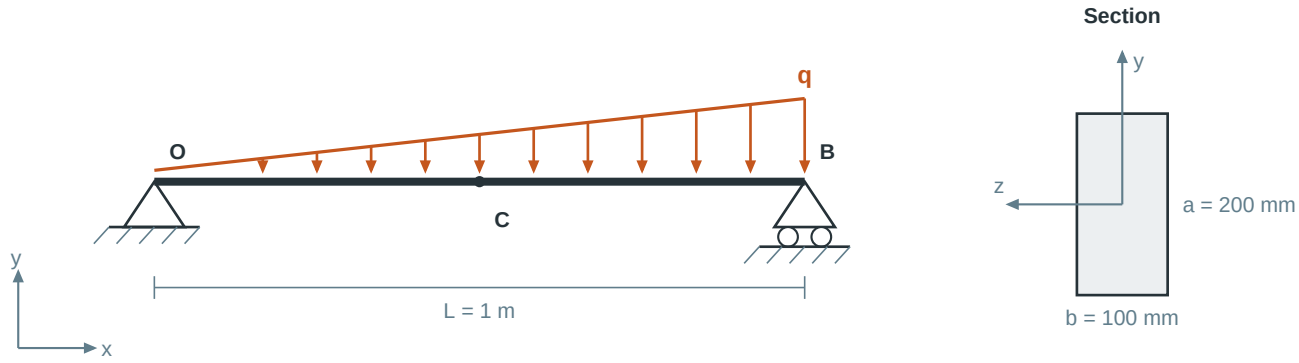
Conclusion

The load starts in the middle of the element and the results are exact, shear included (it adds 3.7 %). A member can therefore carry a load over any part of its length without being cut at the ends of the load.

Case 2.5 — Beam on two simple supports, load varying linearly

What this case proves. A load that grows linearly from zero at O to q at B: its resultant is not at mid-span, so the two reactions and the two rotations differ.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = 0 B: Uy = Uz = 0
Load	q	from 0 at O to 10000 N/m at B, along –y
Mesh		2 elements, 3 nodes (O, C at mid-span, B)

Reference solution (Timoshenko beam)

$$u_y(C) = -\,5\,q\,L^4 / (768\,E\,I_z) - q\,L^2 / (16\,G\,A_s)$$

$$\theta_z(O) = -\,7\,q\,L^3 / (360\,E\,I_z) \qquad \theta_z(B) = 8\,q\,L^3 / (360\,E\,I_z)$$

$$R_y(O) = q\,L / 6 \qquad R_y(B) = q\,L / 3 \qquad \text{in the member at C: } M_z = q\,L^2 / 16, \quad V_y = q\,L / 24$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at C, u_y [m]	$-5\,q\,L^4/(768\,E\,I_z) - q\,L^2/(16\,G\,A_s)$	-5.370312e-06	-5.370312e-06	1.6e-16	OK
Rotation at O, θ_z [rad]	$-7\,q\,L^3/(360\,E\,I_z)$	-1.458333e-05	-1.458333e-05	0.0e+00	OK
Rotation at B, θ_z [rad]	$8\,q\,L^3/(360\,E\,I_z)$	1.666667e-05	1.666667e-05	0.0e+00	OK
Reaction at O, R_y [N]	q L/6	1666.667	1666.667	2.0e-07	OK
Reaction at B, R_y [N]	q L/3	3333.333	3333.333	1.0e-07	OK
Member at C, moment M_z [N.m]	$q\,L^2/16$	625	625	0.0e+00	OK
Member at C, shear V_y [N]	q L/24	416.6667	416.667	8.0e-07	OK
Deflection at C without shear areas [m]	$-5\,q\,L^4/(768\,E\,I_z)$	-4.882812e-06	-4.882812e-06	0.0e+00	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

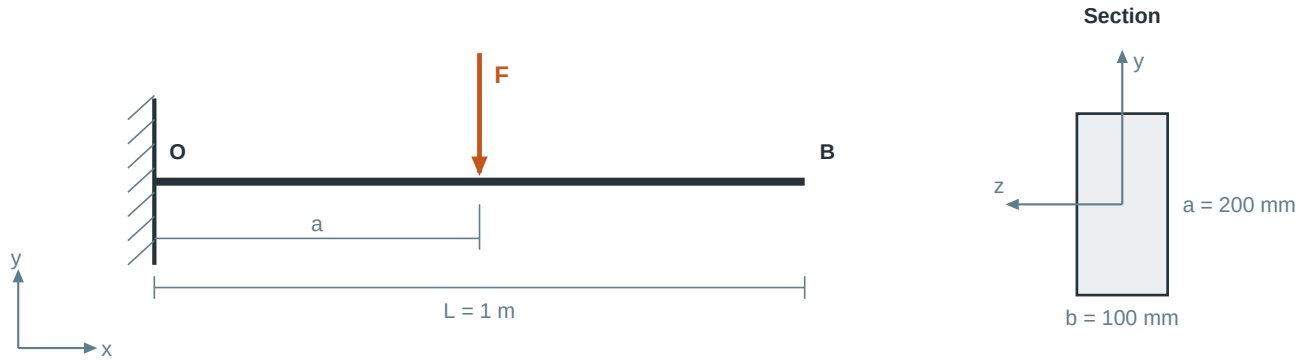
Conclusion

The support under the heavier end takes two thirds of the load, and the rotations are in the ratio 7 to 8. The deflection and the moment at mid-span are half those of case 2.3, as the triangular load and its mirror image add up to the uniform load. Shear adds 10.0 %.

Case 2.6 — Cantilever, force applied between the nodes

What this case proves. A concentrated force applied on a member at a distance a from its first end, where there is no node: the part of the member beyond the force stays straight and unloaded.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F, a	10000 N along –y, on the member at a = 0.500 m from O
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$$u_y(B) = -F a^2 (3L - a) / (6 E I_z) - F a / (G A_s)$$

$$\theta_z(B) = -F a^2 / (2 E I_z) \qquad R_y(O) = F \qquad M_z(O) = F a$$

In the member: $V_y(O) = F$, $M_z(O) = -F a$, $V_y(B) = M_z(B) = 0$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_y [m]	$-F a^2(3L - a)/(6 E I_z) - F a/(G A_s)$	-8.2025e-05	-8.2025e-05	1.7e-16	OK
Rotation at B, θ_z [rad]	$-F a^2/(2 E I_z)$	-9.375e-05	-9.375e-05	2.9e-16	OK
Reaction at O, R_y [N]	F	10000	10000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	F a	5000	5000	0.0e+00	OK
Member at O, shear V_y [N]	F	10000	10000	0.0e+00	OK
Member at O, moment M_z [N.m]	–F a	-5000	-5000	0.0e+00	OK
Member at B, shear V_y [N]	0	0	0	0.0e+00	OK
Member at B, moment M_z [N.m]	0	0	0	0.0e+00	OK
Deflection at B without shear areas [m]	$-F a^2(3L - a)/(6 E I_z)$	-7.8125e-05	-7.8125e-05	1.7e-16	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

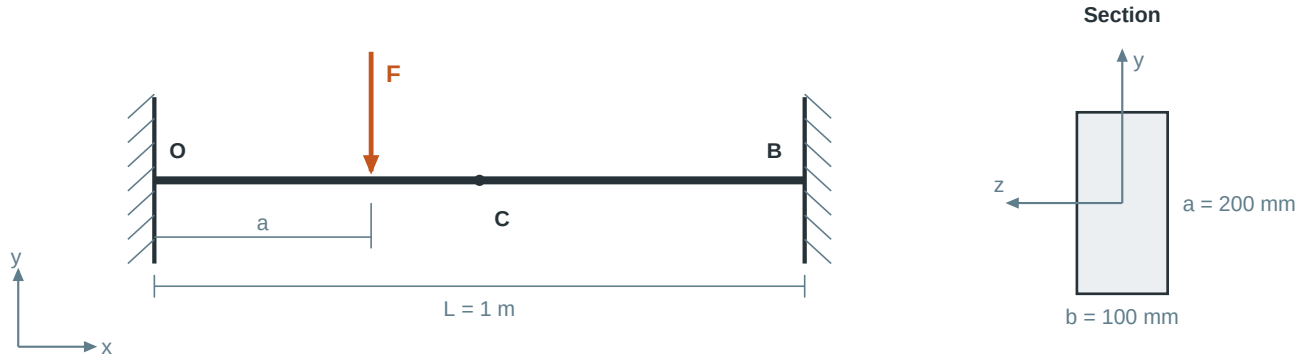
Conclusion

The tip moves by the deflection under the force plus the rotation there times the remaining length, and the shear term is F a / (G A_s), as only the length a carries a shear force (5.0 % more). The end B of the member carries nothing.

Case 2.7 — Beam clamped at both ends, force applied between the nodes

What this case proves. An off-centre force on a statically indeterminate beam. Here shear deformation changes the clamping moments themselves, not only the deflection: the load on the member must be treated with the same shear stiffness as the member.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O and B: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F, a	10000 N along –y, on the first element at a = L/3 from O; b = L – a
Mesh		2 elements, 3 nodes (O, C at mid-span, B)

Reference solution (Timoshenko beam)

$$\Phi = 12\,E\,I_z / (G\,A_s\,L^2) = 0.1248$$

$$M_O = F\,a\,b\,(b + \Phi\,L/2) / [L^2\,(1 + \Phi)] \qquad M_B = F\,a\,b\,(a + \Phi\,L/2) / [L^2\,(1 + \Phi)]$$

$$R_y(O) = F\,b / L + (M_O - M_B) / L \qquad R_y(B) = F - R_y(O)$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Clamping moment at O, M_z [N.m]	M_O	1440.388	1440.388	1.7e-07	OK
Clamping moment at B, M_z [N.m]	$-M_B$	-781.8345	-781.834	6.0e-07	OK
Reaction at O, R_y [N]	F b/L + $(M_O - M_B)/L$	7325.22	7325.22	5.9e-09	OK
Reaction at B, R_y [N]	F – $R_y(O)$	2674.78	2674.78	1.6e-08	OK
Member at O, moment M_z [N.m]	$-M_O$	-1440.388	-1440.388	1.7e-07	OK
Member at B, moment M_z [N.m]	$-M_B$	-781.8345	-781.834	6.0e-07	OK
Clamping moment at O without shear areas [N.m]	F $a\,b^2/L^2$	1481.481	1481.481	3.3e-07	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

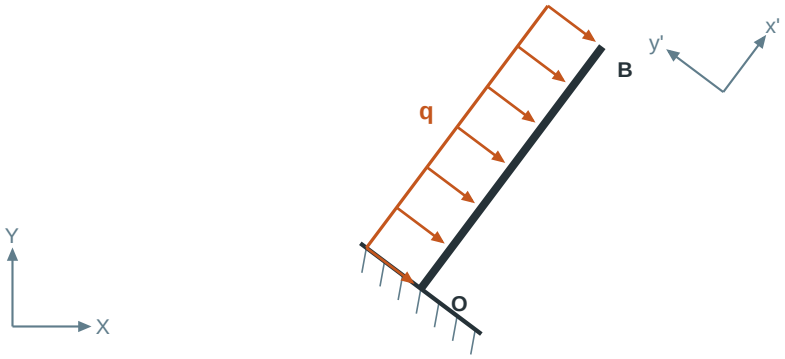
Conclusion

With shear deformation the clamping moment at O is 1440.4 N.m; the Euler-Bernoulli value F a b^2/L^2 is 1481.5 N.m, 2.9 % more. KUSOR gives each value with the corresponding shear areas: the force on the member and the stiffness of the member are consistent. The node C at mid-span carries no load.

Case 2.8 — Inclined member, load given in the axes of the member

What this case proves. Case 2.1 on a member that follows no axis of the model, the load being given across the member (along –y'): the member bends as in case 2.1 and carries no axial force.

Model



Member O B

L = 1 m, along (0.6, 0.8, 0)

Section of case 2.1, side a along y'

Load q

across the member, along -y'

per unit length of the member

Data	Symbol	Value
Member	L, x'	1.000 m, along x' = (0.6, 0.8, 0) in the axes of the model
Axes of the section	y', z'	y' = (–0.8, 0.6, 0) z' = (0, 0, 1)
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	q	10000 N/m in the axes of the member, along –y'
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$u_{y'}(B) = – q L^4 / (8 E I_z) – q L^2 / (2 G A_s)$ (as in case 2.1)

$U(B) = u_{y'}\, y'$ $R(O) = q L\, y' = (–0.8\, q L, 0.6\, q L, 0)$ $M_z(O) = q L^2 / 2$

In the member at O: N = 0, $V_y = q L$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B along y' [m]	$–q L^4/(8 E I_z) – q L^2/(2 G A_s)$	-9.765e-05	-9.765e-05	4.0e-15	OK
Displacement at B, U_x [m]	$–0.8\, u_{y'}$	7.812e-05	7.812e-05	4.0e-15	OK
Displacement at B, U_y [m]	$0.6\, u_{y'}$	-5.859e-05	-5.859e-05	4.2e-15	OK
Reaction at O, R_x [N]	–0.8 q L	-8000	-8000	0.0e+00	OK
Reaction at O, R_y [N]	0.6 q L	6000	6000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	$q L^2/2$	5000	5000	0.0e+00	OK
Member at O, axial force N [N]	0	0	0	0.0e+00	OK
Member at O, shear V_y [N]	q L	10000	10000	0.0e+00	OK
Deflection along y', no shear areas [m]	$–q L^4/(8 E I_z)$	-9.375e-05	-9.375e-05	8.7e-16	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

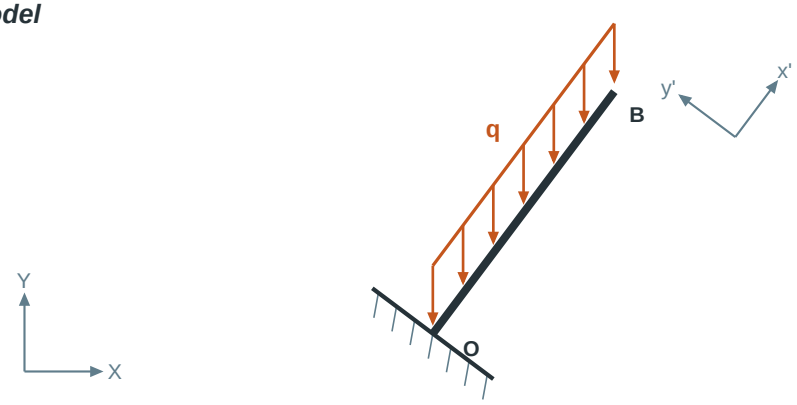
Conclusion

In the axes of the member the result is that of case 2.1; in the axes of the model the displacement and the reaction are its components along y'. A load given in the axes of a member follows the member.

Case 2.9 — Inclined member, load given in the axes of the model

What this case proves. The same member under a vertical load given in the axes of the model (a weight): one part of the load acts across the member and bends it, the other acts along it and compresses it.

Model



Member O B

L = 1 m, along (0.6, 0.8, 0)

Section of case 2.1, side a along y'

Load q

vertical, along -Y of the model

per unit length of the member

Data	Symbol	Value
Member	L, x'	1.000 m, along x' = (0.6, 0.8, 0) in the axes of the model
Axes of the section	y', z'	y' = (–0.8, 0.6, 0) z' = (0, 0, 1)
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	$I_z = b\ a^3/12$	6.6667e-05 m^4
Shear area	$A_s = 5/6\ A$	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	q	10000 N/m along –Y, per unit length of the member
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

Across the member: 0.6 q along –y' along the member: 0.8 q along –x'

$u_{y'}(B) = -0.6\ q\ [L^4 / (8\ E\ I_z) + L^2 / (2\ G\ A_s)]$ $u_{x'}(B) = -0.8\ q\ L^2 / (2\ E\ A)$

$R_y(O) = q\ L$ $M_z(O) = q\ L\ (0.6\ L / 2)$ in the member at O: N = – 0.8 q L, $V_y = 0.6\ q\ L$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B along y' [m]	0.6 times that of case 2.8	-5.859e-05	-5.859e-05	4.5e-15	OK
Shortening at B along x' [m]	$-0.8\ q\ L^2/(2\ E\ A)$	-1e-06	-1e-06	1.5e-15	OK
Reaction at O, R_x [N]	0	0	0	0.0e+00	OK
Reaction at O, R_y [N]	q L	10000	10000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	$0.6\ q\ L^2/2$	3000	3000	0.0e+00	OK
Member at O, axial force N [N]	–0.8 q L	-8000	-8000	0.0e+00	OK
Member at O, shear V_y [N]	0.6 q L	6000	6000	0.0e+00	OK
Member at O, moment M_z [N.m]	$-0.6\ q\ L^2/2$	-3000	-3000	0.0e+00	OK
Deflection along y', no shear areas [m]	$-0.6\ q\ L^4/(8\ E\ I_z)$	-5.625e-05	-5.625e-05	6.0e-16	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

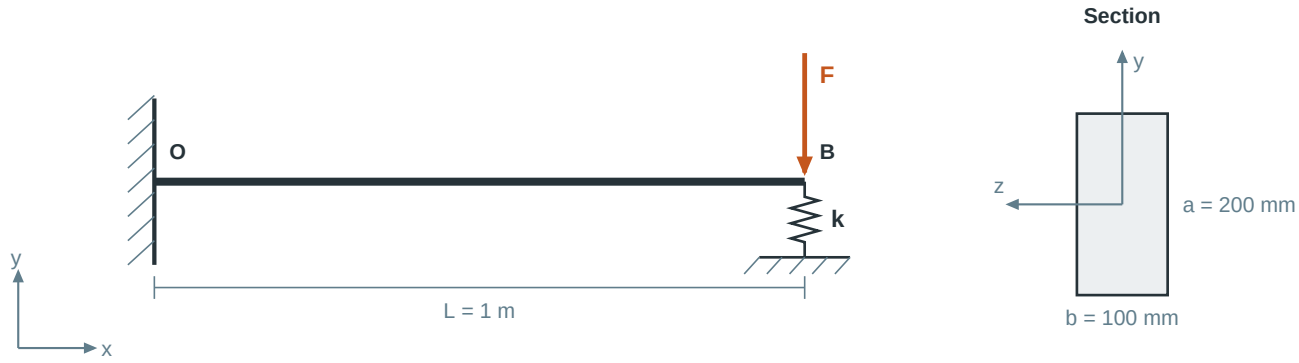
Conclusion

The reaction is vertical and equal to the whole load q L, and the clamping moment is that load times the horizontal distance to its centre. The axial force is negative: compression.

Case 3.1 — Cantilever with its tip on a spring

What this case proves. An elastic support in translation: the force is shared between the spring and the beam in proportion to their stiffnesses.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0 B: spring k = 2e+07 N/m along Uy
Load	F	10000 N at B, along –y
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$f = L^3 / (3 E I_z) + L / (G A_s)$ (deflection of the tip of the beam alone, per unit force)

$u_y(B) = – F / (k + 1/f)$ force in the spring = $k\,|u_y(B)|$

$R_y(O) = F – k\,|u_y(B)|$ $M_z(O) = R_y(O)\,L$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_y [m]	–F/(k + 1/f)	-0.0001700977	-0.0001700977	4.8e-16	OK
Force in the spring at B [N]	$k\,\lvert u_y\rvert$	3401.953	3401.953	6.4e-09	OK
Reaction at O, R_y [N]	$F – k\,\lvert u_y\rvert$	6598.047	6598.047	3.3e-09	OK
Clamping moment at O, M_z [N.m]	$(F – k\,\lvert u_y\rvert)\,L$	6598.047	6598.047	3.3e-09	OK
Deflection at B without shear areas [m]	$–F/(k + 3 E I_z/L^3)$	-0.0001666667	-0.0001666667	0.0e+00	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

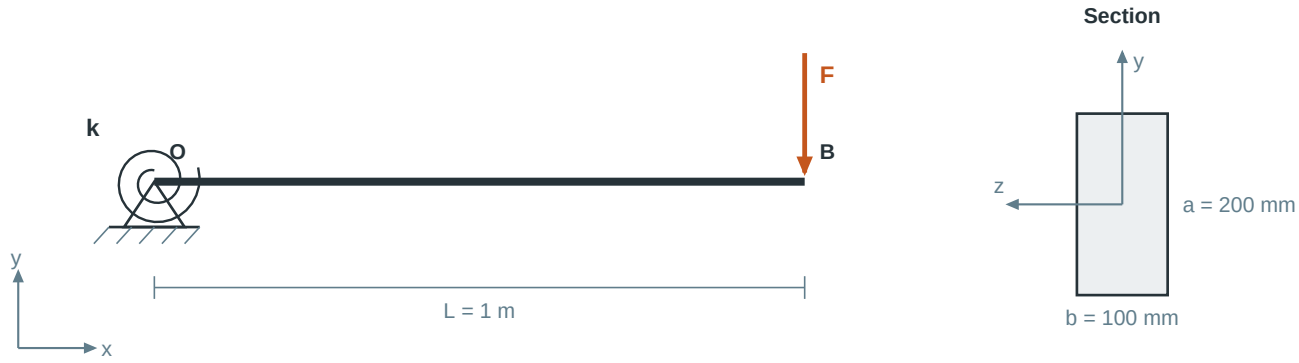
Conclusion

The spring takes 3402 N and the beam 6598 N, that is 34 % and 66 % of the force. The force in the spring is given with the reactions, at the node that carries the spring. The stiffness of the beam includes its shear flexibility: without it the share of the spring would be different.

Case 3.2 — Cantilever on a support that is elastic in rotation

What this case proves. An elastic support in rotation: the support turns under the clamping moment, and that rotation carries the whole beam with it.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = 0, spring k = 2e+07 N.m/rad about Rz
Load	F	10000 N at B, along –y
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$u_y(B) = -F L^3 / (3 E I_z) - F L / (G A_s) - F L^2 / k$ (bending, shear, rotation of the support)

$\theta_z(O) = -F L / k$ $\theta_z(B) = \theta_z(O) - F L^2 / (2 E I_z)$

$R_y(O) = F$ moment in the spring = F L

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_y [m]	see the reference solution	-0.0007578	-0.0007578	1.4e-16	OK
Rotation at O, θ_z [rad]	–F L/k	-0.0005	-0.0005	4.3e-16	OK
Rotation at B, θ_z [rad]	$-F L/k - F L^2/(2 E I_z)$	-0.000875	-0.000875	2.5e-16	OK
Reaction at O, R_y [N]	F	10000	10000	0.0e+00	OK
Moment in the spring at O, M_z [N.m]	F L	10000	10000	0.0e+00	OK
Deflection at B without shear areas [m]	bending and spring terms	-0.00075	-0.00075	0.0e+00	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

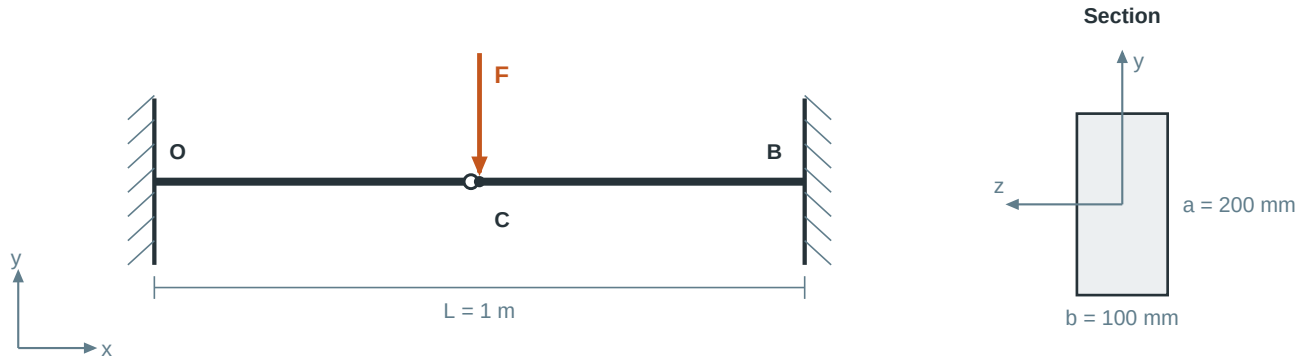
Conclusion

The rotation of the support, F L / k = 0.0005 rad, adds 0.0005 m at the tip: more than the deformation of the beam itself (0.0002578 m). A rotation is not held at O, the spring alone carries the moment F L.

Case 3.3 — Beam clamped at both ends with a hinge at mid-span

What this case proves. A hinge inside a structure, made by releasing the rotation of a member at one of its ends: the bending moment is zero at the hinge and each half works as a cantilever carrying F/2.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O and B: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F	10000 N at C (mid-span), along –y
Mesh		2 elements, 3 nodes; element 1 (O to C) is released in rotation Rz at its end C

Reference solution (Timoshenko beam)

$$u_y(C) = -(F/2)\,[(L/2)^3 / (3\,E\,I_z) + (L/2) / (G\,A_s)] = -F\,L^3 / (48\,E\,I_z) - F\,L / (4\,G\,A_s)$$

$$M_z(O) = F\,L / 4 = -M_z(B) \qquad M_z = 0 \text{ in both members at C}$$

$$\theta_z(C) = F\,L^2 / (16\,E\,I_z) \quad \text{(the node turns with member 2, which is not released)}$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at C, u_y [m]	$-F\,L^3/(48\,E\,I_z) - F\,L/(4\,G\,A_s)$	-1.7575e-05	-1.7575e-05	1.9e-16	OK
Clamping moment at O, M_z [N.m]	F L/4	2500	2500	0.0e+00	OK
Clamping moment at B, M_z [N.m]	–F L/4	-2500	-2500	0.0e+00	OK
Reaction at O, R_y [N]	F/2	5000	5000	0.0e+00	OK
Member 1 at O, moment M_z [N.m]	–F L/4	-2500	-2500	0.0e+00	OK
Member 1 at C, moment M_z [N.m]	0	0	0	0.0e+00	OK
Member 2 at C, moment M_z [N.m]	0	0	0	0.0e+00	OK
Rotation at C, θ_z [rad]	$F\,L^2/(16\,E\,I_z)$	4.6875e-05	4.6875e-05	1.4e-16	OK
Deflection at C without shear areas [m]	$-F\,L^3/(48\,E\,I_z)$	-1.5625e-05	-1.5625e-05	0.0e+00	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

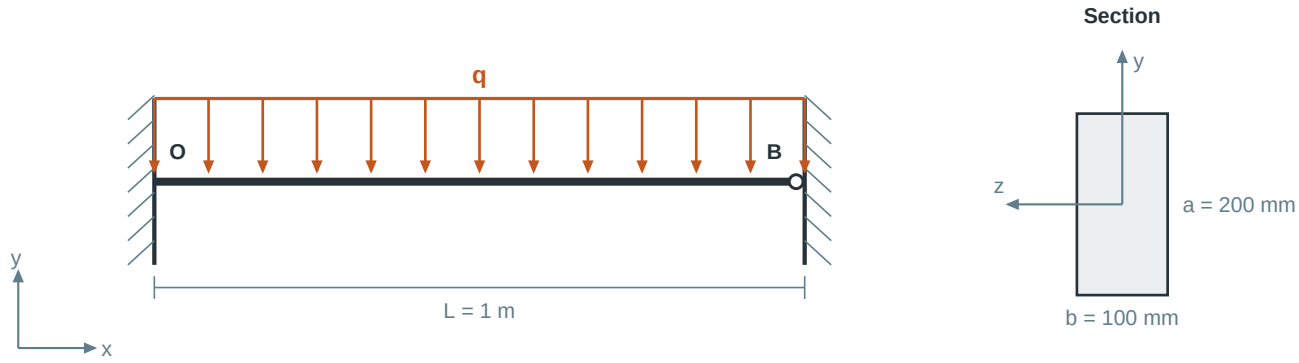
Conclusion

The moment is zero on both sides of the hinge and the clamping moments are F L / 4, twice those of case 1.8 without the hinge. The deflection is that of the beam on simple supports of case 1.7. The node C keeps the rotation of the member that is still attached to it; the released end turns freely.

Case 3.4 — Released member under a uniform load

What this case proves. A hinge and a load on the same member. The support B holds the rotation, but the member is released there: the beam is clamped at O and simply propped at B, and the load on the member must follow.

Model



Data	Symbol	Value
Length	L	1.000 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0 B: Uy = Uz = Rx = Ry = Rz = 0
Load	q	10000 N/m over the whole member, along –y
Mesh		1 element, 2 nodes; released in rotation Rz at its end B

Reference solution (Timoshenko beam)

$$R_B = [q\,L^4 / (8\,E\,I_z) + q\,L^2 / (2\,G\,A_s)] / [L^3 / (3\,E\,I_z) + L / (G\,A_s)] \quad \text{(no deflection at B)}$$

$$R_y(O) = q\,L - R_B \qquad M_z(O) = q\,L^2 / 2 - R_B\,L \qquad M_z(B) = 0$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Reaction at B, R_y [N]	R_B	3787.82	3787.82	4.1e-09	OK
Reaction at O, R_y [N]	q L – R_B	6212.18	6212.18	2.5e-09	OK
Clamping moment at O, M_z [N.m]	q L^2/2 – R_B L	1212.18	1212.18	1.3e-08	OK
Moment at the support B, M_z [N.m]	0	0	0	0.0e+00	OK
Member at O, moment M_z [N.m]	–(q L^2/2 – R_B L)	-1212.18	-1212.18	1.3e-08	OK
Member at B, moment M_z [N.m]	0	0	0	0.0e+00	OK
Member at B, shear V_y [N]	–R_B	-3787.82	-3787.82	4.1e-09	OK
Reaction at B without shear areas [N]	3 q L/8	3750	3750	0.0e+00	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

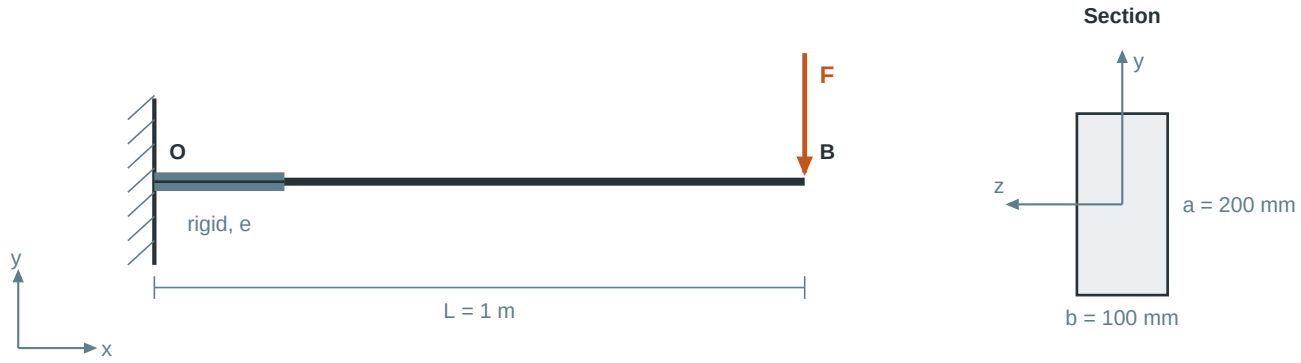
Conclusion

No moment reaches the support B although it holds the rotation, and the reaction there is 3787.8 N; the Euler-Bernoulli value 3 q L / 8 is 3750 N. The load on a released member is therefore applied to the member as it really is, hinge and shear deformation included.

Case 3.5 — Member offset along its axis, at the clamp

What this case proves. An offset along the member, which is a rigid arm: the member starts at a distance e from its node, the length e being rigid. The flexible length is L' = L – e.

Model



Data	Symbol	Value
Length	L	1.000 m
Offset	e	0.200 m from O, along x; flexible length L' = L – e = 0.800 m
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F	10000 N at B, along –y
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$$u_y(B) = -F L'^3 / (3 E I_z) - F L' / (G A_s) \quad \text{(the cantilever of length L')}$$

$$\theta_z(B) = -F L'^2 / (2 E I_z) \qquad M_z(O) = F L \qquad M_z \text{ at the first end of the member} = -F L'$$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_y [m]	$-F L'^3/(3 E I_z) - F L'/(G A_s)$	-0.00013424	-0.00013424	0.0e+00	OK
Rotation at B, θ_z [rad]	$-F L'^2/(2 E I_z)$	-0.00024	-0.00024	1.1e-16	OK
Reaction at O, R_y [N]	F	10000	10000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	F L	10000	10000	0.0e+00	OK
Member at its first end, shear V_y [N]	F	10000	10000	0.0e+00	OK
Member at its first end, moment M_z [N.m]	–F L'	-8000	-8000	0.0e+00	OK
Deflection at B without shear areas [m]	$-F L'^3/(3 E I_z)$	-0.000128	-0.000128	2.1e-16	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

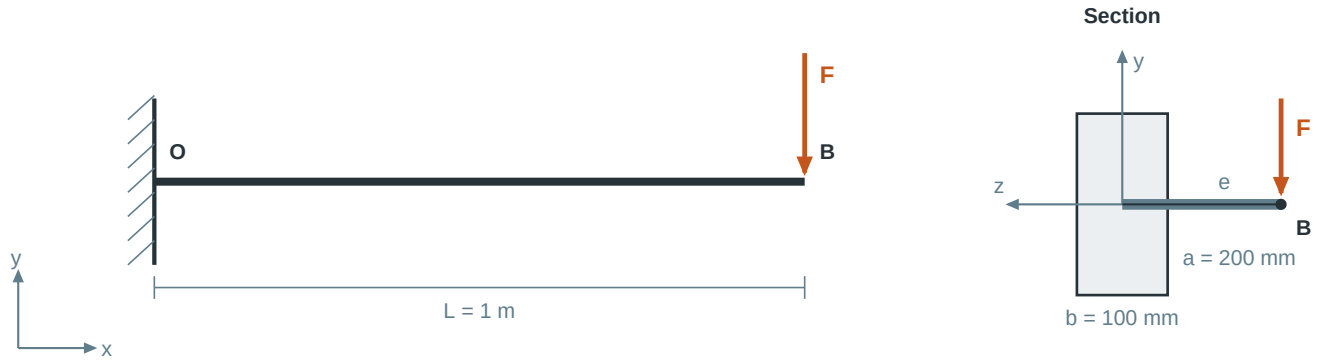
Conclusion

The beam deforms over L' only, so the deflection is 52 % of that of case 1.1. The moment at the support is still F L, the arm carrying the force to the node; the moment given for the member is the one at the start of its flexible length, F L', where its section is checked.

Case 3.6 — Member offset across its axis, at the loaded end

What this case proves. An offset across the member, at one end only: the node B is at a distance e from the axis of the member, along –z, and the force applied at B bends the member and twists it.

Model



Data	Symbol	Value
Length	L	1.000 m
Offset	e	0.200 m, from the end of the member to the node B, along –z
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Torsion constant	J = 0.229 a b^3	4.58e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F	10000 N at the node B, along –y
Mesh		1 element, 2 nodes

Reference solution (Timoshenko beam)

$u_y(B) = – F L^3 / (3 E I_z) – F L / (G A_s) – F e^2 L / (G J)$ (bending, shear, twist times the arm)

$\theta_x(B) = – F e L / (G J)$ $\theta_z(B) = – F L^2 / (2 E I_z)$

$M_x(O) = F e$ $M_z(O) = F L$ torque in the member = F e

Results

Quantity	Formula	Reference	KUSOR	Difference	
Deflection at B, u_y [m]	see the reference solution	-0.0003713371	-0.0003713371	1.5e-16	OK
Twist at B, θ_x [rad]	–F e L/(G J)	-0.0005676856	-0.0005676856	1.9e-16	OK
Rotation at B, θ_z [rad]	$–F L^2/(2 E I_z)$	-0.000375	-0.000375	4.3e-16	OK
Reaction at O, M_x [N.m]	F e	2000	2000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	F L	10000	10000	0.0e+00	OK
Member, torque |T| [N.m]	F e	2000	2000	0.0e+00	OK
Deflection at B without shear areas [m]	bending and twist terms	-0.0003635371	-0.0003635371	0.0e+00	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

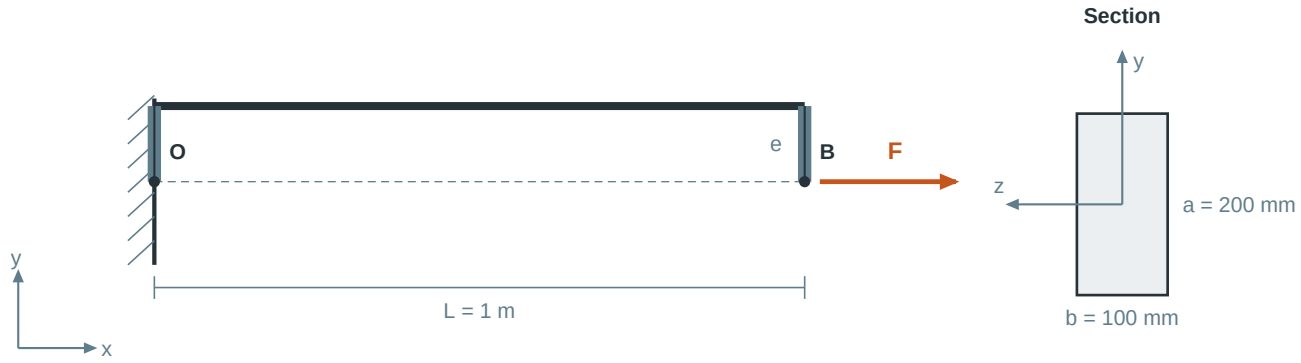
Conclusion

The eccentric force gives a torque F e = 2000 N.m in the member; its twist lowers the node B by a further 0.0001135 m, to be added to the 0.0002578 m of case 1.1. The rigid arm transmits the displacements and the rotations of the end of the member to the node exactly.

Case 3.7 — Member offset to one side of its nodes, under an axial force

What this case proves. An offset across the member at both ends: the axis of the member is parallel to the line of its nodes, at a distance e. A force along the line of the nodes pulls the member and bends it.

Model



Data	Symbol	Value
Length	L	1.000 m
Offset	e	0.200 m along +y, at both ends of the member
Section	rectangle	a = 200 mm (along y), b = 100 mm (along z)
Area	A = a b	0.02 m^2
Second moment of area	I_z = b a^3/12	6.6667e-05 m^4
Shear area	A_s = 5/6 A	0.016667 m^2
Material	E, ν	E = 2e+11 Pa, ν = 0.3, G = E/[2(1+ν)] = 7.6923e+10 Pa
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F	10000 N at the node B, along +x
Mesh		1 element, 2 nodes

Reference solution

In the member: N = F and M_z = F e, constant along its length

$u_x(B) = F\,L / (E\,A) + F\,e^2\,L / (E\,I_z)$ (stretching, then the rotation of the end times the offset)

$u_y(B) = F\,e\,L^2 / (2\,E\,I_z)$ $\theta_z(B) = F\,e\,L / (E\,I_z)$ $M_z(O) = 0$

Results

Quantity	Formula	Reference	KUSOR	Difference	
Displacement at B, u_x [m]	$F\,L/(E\,A) + F\,e^2\,L/(E\,I_z)$	3.25e-05	3.25e-05	1.9e-15	OK
Displacement at B, u_y [m]	$F\,e\,L^2/(2\,E\,I_z)$	7.5e-05	7.5e-05	1.8e-15	OK
Rotation at B, θ_z [rad]	$F\,e\,L/(E\,I_z)$	0.00015	0.00015	1.8e-15	OK
Reaction at O, R_x [N]	–F	-10000	-10000	0.0e+00	OK
Clamping moment at O, M_z [N.m]	0	0	0	0.0e+00	OK
Member, axial force N [N]	F	10000	10000	0.0e+00	OK
Member at its first end, moment M_z [N.m]	F e	2000	2000	0.0e+00	OK
Member at its second end, moment M_z [N.m]	F e	2000	2000	0.0e+00	OK

Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

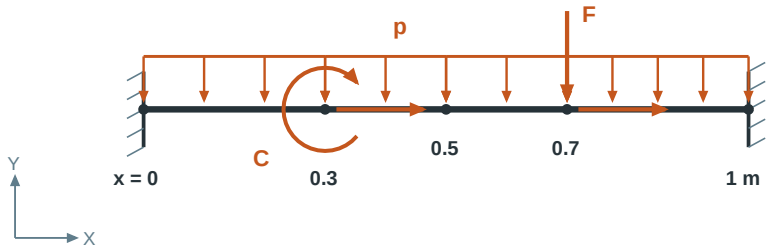
Conclusion

The offset turns the axial force into a constant moment F e = 2000 N.m in the member, which the member forces show at both ends. The node B moves 13 times further along x than the stretching alone would give. No moment reaches the support, the force passing through the node O. There is no shear force here, so shear deformation plays no part.

Case 4.1 — SSLL01 — Slender beam clamped at both ends

What this case proves. Several loads of different kinds on one beam, a load per unit length, a couple, a transverse force and two axial forces, with the forces inside the beam at mid-span.

Model



Beam of 1 m, along X
p over the whole span
couple C at x = 0.3 m
force F at x = 0.7 m
axial forces at 0.3 and 0.7 m

Data	Symbol	Value
Length	L	1 m, nodes at x = 0, 0.3, 0.5, 0.7 and 1 m
Section	I, A	I = 1.7e-8 m^4, no shear deformation; A = 1e-3 m^2
Material	E	2.1e11 Pa
Supports		x = 0 and x = 1 m: Ux = Uy = Uz = Rx = Ry = Rz = 0
Loads	p, C, F	p = 24 000 N/m along –Y; C = –3 000 N.m at 0.3 m; F = 20 000 N along –Y at 0.7 m; 30 000 N and 10 000 N along +X at 0.3 and 0.7 m
Mesh		4 elements, 5 nodes

Reference solution (code_aster V3.01.001, from the AFNOR guide)

The sheet gives the loads on a figure; they are read from it, and from the nodal couples it lists.

AFNOR prints the moment and the deflection with two figures (2 800 N.m, 49 mm). The same quantities are therefore also computed here by integrating the elastic line of the clamped beam.

The sheet gives no area; it does not affect the quantities checked.

Results

Quantity	Origin of the reference	Reference	KUSOR	Difference	
Shear at mid-span [N]	AFNOR, two figures	540	540	0.0e+00	OK
Moment at mid-span [N.m]	AFNOR, two figures	2800	2800	0.0e+00	OK
Deflection at mid-span [m]	AFNOR, two figures	-0.049	-0.04901961	4.0e-04	OK
Axial reaction at x = 0 [N]	published by AFNOR	24000	24000	0.0e+00	OK
Shear at mid-span [N]	elastic line integrated	539.82	540	3.3e-04	OK
Moment at mid-span [N.m]	elastic line integrated	2800.075	2800	2.7e-05	OK
Deflection at mid-span [m]	elastic line integrated	-0.04901908	-0.04901961	1.1e-05	OK

Pass criterion: relative difference ≤ 5e-04; ≤ 1e-02 for the values AFNOR prints with two figures.

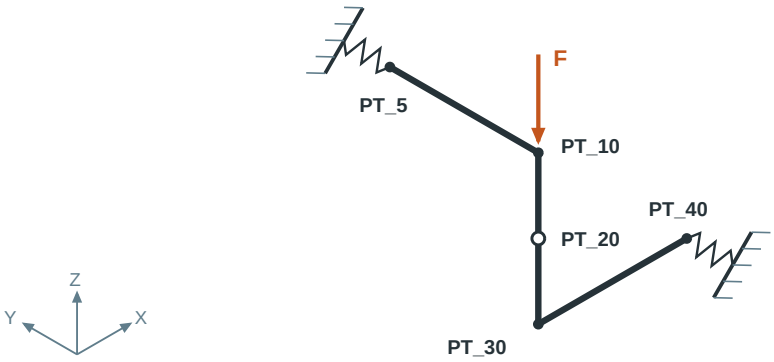
Conclusion

The four published values are met within their rounding, and the values recomputed by hand to better than 0.05 %. The axial reaction shows that the axial forces are shared between the two clamps independently of the bending loads.

Case 4.2 — SSLL04 — Space structure on elastic supports

What this case proves. A structure in three dimensions: members along the three axes, a ball joint between two of them, and supports that are elastic in translation and in rotation.

Model



Three members of 2 m

PT_5 - PT_10 along Y

PT_10 - PT_30 along Z, with a ball joint at its middle PT_20

PT_30 - PT_40 along X

Elastic supports at PT_5, PT_40

Data	Symbol	Value
Geometry		PT_5 (0, 2, 2), PT_10 (0, 0, 2), PT_20 (0, 0, 1), PT_30 (0, 0, 0), PT_40 (2, 0, 0) m
Section	A, I, J	A = 1e-3 m^2, $I_y = I_z$ = 1e-6 m^4, J = 2e-6 m^4, no shear deformation
Material	E, G	E = 2.1e11 Pa, G = 7.875e10 Pa
Supports		PT_5: Ux = Uz = Ry = 0, springs on Uy, Rx, Rz PT_40: Uy = Uz = Rx = 0, springs on Ux, Ry, Rz
Springs	k	52 500 N/m in translation, 52 500 N.m/rad in rotation
Load	F	10 000 N at PT_10, along –Z
Mesh		4 elements; the element PT_10 - PT_20 is released in its three rotations at PT_20

Reference solution (code_aster V3.01.004, from the AFNOR guide)

The geometry and the direction of the load are rebuilt from the figure and the supports of the sheet.

G = 78 750 MPa is used as given (a Poisson ratio of 1/3, not the 0.30 written on the sheet).

The sheet mentions self-weight without a density: it is left out. Values compared in absolute value.

Results

Quantity	Origin of the reference	Reference	KUSOR	Difference	
|Mx| at PT_5 [N.m]	published by AFNOR	8437.5	8438.141	7.6e-05	OK
|torsion| at PT_5 [N.m]	published by AFNOR	1562.5	1562.302	1.3e-04	OK
|Mz| at PT_5 [N.m]	published by AFNOR	3125	3124.604	1.3e-04	OK
|torsion| at PT_40 [N.m]	published by AFNOR	1562.5	1562.503	1.9e-06	OK
|My| at PT_40 [N.m]	published by AFNOR	8437.5	8437.055	5.3e-05	OK
|Mz| at PT_40 [N.m]	published by AFNOR	3125	3125.006	1.9e-06	OK
|displacement| at PT_5 [m]	published by AFNOR	0.029762	0.02976196	1.4e-06	OK
|rotation| at PT_5 [rad]	published by AFNOR	0.16071	0.1607265	1.0e-04	OK
|displacement| at PT_20 [m]	published by AFNOR	0.37004	0.3700443	1.2e-05	OK

Pass criterion: relative difference ≤ 5e-04, the published values being printed with five to eight figures.

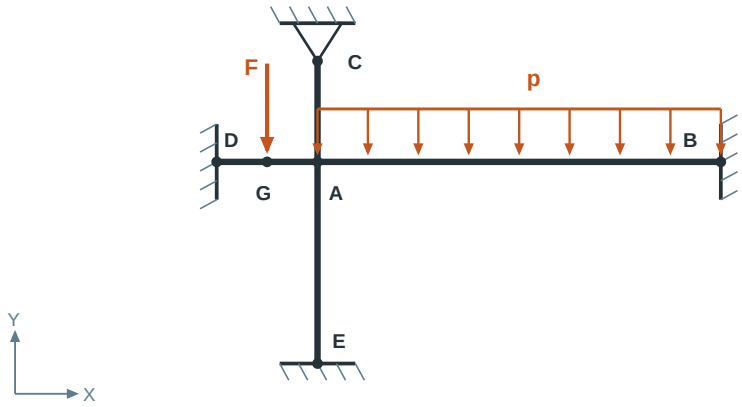
Conclusion

The nine published values are met: moments in the elastic supports, displacement and rotation of a support, displacement of the ball joint. Springs, releases and members in the three directions work together.

Case 4.3 — SSLL10 — Frame with lateral links

What this case proves. The sharing of a moment between four members of different stiffness that meet at one node, which can only turn.

Model



Four members meeting at A

AB = 4 m, AC = 1 m

AD = 1 m, AE = 2 m

G: middle of AD

B, D, E clamped; C pinned

Data	Symbol	Value
Geometry		A (0, 0), B (4, 0), C (0, 1), D (–1, 0), E (0, –2) m; G middle of AD
Sections	I	AB: 64/3 e-8 m^4 AC and AD: 1/12 e-8 m^4 AE: 4/3 e-8 m^4; no shear deformation, very large area
Material	E	2.0e11 Pa
Supports		B, D, E: Ux = Uy = Uz = Rx = Ry = Rz = 0 C: Ux = Uy = Uz = Rx = Ry = 0
Loads	F, p	F = 100 000 N at G, along –Y; p = 1 000 N/m on AB, along –Y
Mesh		5 elements (AD is cut at G), 6 nodes

Reference solution (code_aster V3.01.010, from the AFNOR guide)

The text of the sheet puts the load p on AD; its figure, its formulas and its model put it on AB: AB is used.

The reference keeps bending only: the members are given a very large area and no shear deformation.

Values compared in absolute value, the sign conventions of the two programs being different.

Results

Quantity	Origin of the reference	Reference	KUSOR	Difference	
|θ(A)| [rad]	published, analytical	0.227118	0.2271186	2.8e-06	OK
|M_AB| at A [N.m]	published, analytical	11023.72	11023.73	7.3e-07	OK
|M_AC| at A [N.m]	published, analytical	113.559	113.559	0.0e+00	OK
|M_AD| at A [N.m]	published, analytical	12348.59	12348.59	0.0e+00	OK
|M_AE| at A [N.m]	published, analytical	1211.299	1211.299	3.3e-07	OK

Pass criterion: relative difference ≤ 5e-04, the published values being printed with five to eight figures.

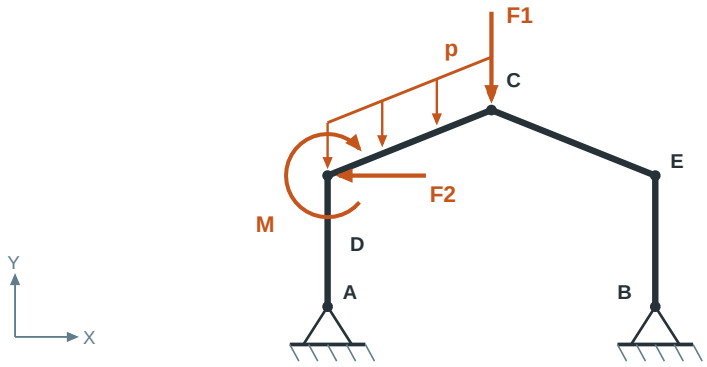
Conclusion

The rotation of the node A and the moments at A in the four members are those of the sheet: the moment is shared between the members in proportion to their stiffness, with a load on one member and a force on another.

Case 4.4 — SSLL14 — Portal frame pinned at its feet

What this case proves. A plane frame with sloping members under four loads applied one at a time: a load per unit length on a rafter, a vertical force, a horizontal force and a moment.

Model



Portal frame, 20 m wide
columns AD and BE: 8 m
ridge C at 12 m
A and B pinned
Four loads, one at a time

Data	Symbol	Value
Geometry		A (0, 0), D (0, 8), C (10, 12), E (20, 8), B (20, 0) m
Sections	I	columns: 5e-4 m^4 rafters: 2.5e-4 m^4; no shear deformation, very large area
Material	E	2.1e11 Pa
Supports		A and B: Ux = Uy = Uz = Rx = Ry = 0 (pinned)
Loads	p, F1, F2, M	p = 3 000 N/m on DC, along –Y; F1 = 20 000 N at C, along –Y; F2 = 10 000 N at D, along –X; M = –100 000 N.m at D
Mesh		4 elements, 5 nodes

Reference solution (code_aster V3.01.014, analytical)

The sheet writes F1 = 2 000 N; its own reaction F_y(A) = 10 000 N requires 20 000 N, which is used.

p is counted per unit length of the rafter (the published reaction 24 233 N says so). Moments and reactions are compared in absolute value.

Results

Quantity	Load	Reference	KUSOR	Difference	
Dx(C) [m]	p on DC	0.0110476	0.01104762	1.7e-06	OK
Dy(C) [m]	p on DC	-0.01242237	-0.01242238	1.1e-07	OK
|Mz(C)| [N.m]	p on DC	18672.99	18672.99	0.0e+00	OK
|Fx(A)| [N]	p on DC	5175.37	5175.373	5.8e-07	OK
|Fy(A)| [N]	p on DC	24233.24	24233.24	8.3e-08	OK
Dx(C) [m]	F1 at C	0	0	0.0e+00	OK
Dy(C) [m]	F1 at C	-0.0149733	-0.0149733	7.9e-08	OK
|Mz(C)| [N.m]	F1 at C	41422.16	41422.16	0.0e+00	OK
|Fx(A)| [N]	F1 at C	4881.487	4881.487	0.0e+00	OK
|Fy(A)| [N]	F1 at C	10000	10000	0.0e+00	OK
Dx(C) [m]	F2 at D	-0.03000956	-0.03000956	3.4e-08	OK
Dy(C) [m]	F2 at D	-0.00299466	-0.002994659	1.8e-07	OK
|Mz(C)| [N.m]	F2 at D	8284.432	8284.432	0.0e+00	OK
|Fx(A)| [N]	F2 at D	5976.297	5976.297	0.0e+00	OK
|Fy(A)| [N]	F2 at D	4000	4000	0.0e+00	OK
Dx(C) [m]	M at D	0.0273532	0.02735322	6.8e-07	OK
Dy(C) [m]	M at D	-0.001215646	-0.001215646	1.5e-07	OK
|Mz(C)| [N.m]	M at D	4916.724	4916.724	0.0e+00	OK
|Fx(A)| [N]	M at D	4576.394	4576.394	0.0e+00	OK
|Fy(A)| [N]	M at D	5000	5000	0.0e+00	OK

Pass criterion: relative difference ≤ 5e-04, the published values being printed with five to eight figures.

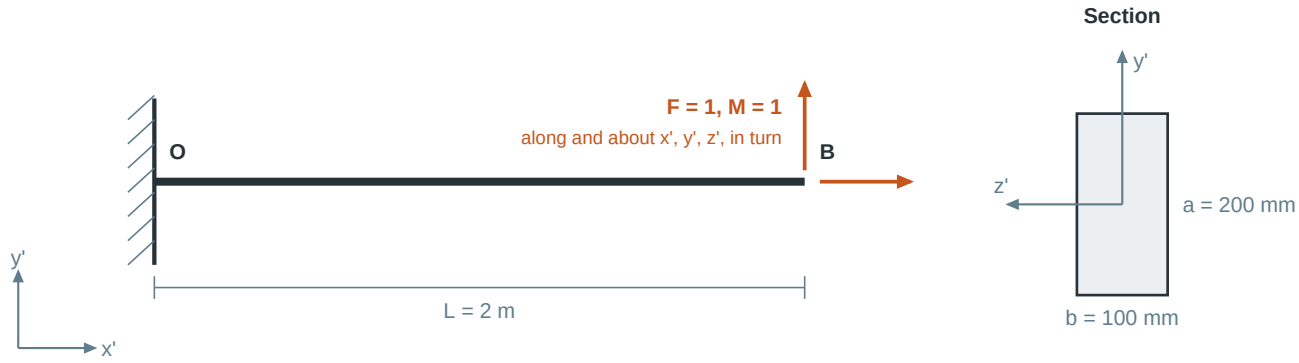
Conclusion

The twenty published values are met: displacements of the ridge, moment at the ridge and reactions at a foot, for each of the four loads.

Case 4.5 — SSLL102 — Clamped beam under unit loads

What this case proves. The six stiffnesses of the beam, with shear deformation, on a member along (1, 1, 1), for a rectangular and a circular section: the published counterpart of part 1.

Model



Data	Symbol	Value
Member	L	2 m, from O along (1, 1, 1)/√3; y' = (–1, 1, 0)/√2
Rectangle	a, b	0.2 m (along y') by 0.1 m, shear factor 5/6, J = 0.229 a b^3
Solid circle	R	0.1 m, shear factor 0.9
Material	E, ν	E = 2.0e11 Pa, ν = 0.3
Supports		O: Ux = Uy = Uz = Rx = Ry = Rz = 0
Loads		at B: a unit force, then a unit moment, along each axis of the member
Mesh		1 element, 2 nodes

Reference solution (code_aster V3.01.102, analytical)

Under a force: $u = L^3(4 + \Phi) / (12\ E\ I)$, $\theta = L^2 / (2\ E\ I)$, with $\Phi = 12\ E\ I / (L^2\ G\ A_s)$

The equal angle of the sheet is left out: its shear centre is not at its centroid.

Results

Quantity	Section and load	Reference	KUSOR	Difference	
$u_{x'}$ at B	rectangle, Fx = 1	5e-10	5e-10	0.0e+00	OK
$u_{y'}$ at B	rectangle, Fy = 1	2.0156e-07	2.0156e-07	2.7e-14	OK
$\theta_{z'}$ at B	rectangle, Fy = 1	1.5e-07	1.5e-07	2.7e-14	OK
$u_{z'}$ at B	rectangle, Fz = 1	8.0156e-07	8.0156e-07	4.5e-14	OK
$\theta_{y'}$ at B	rectangle, Fz = 1	-6e-07	-6e-07	4.5e-14	OK
$\theta_{x'}$ at B	rectangle, Mx = 1	5.676856e-07	5.676856e-07	3.7e-16	OK
$u_{z'}$ at B	rectangle, My = 1	-6e-07	-6e-07	4.5e-14	OK
$\theta_{y'}$ at B	rectangle, My = 1	6e-07	6e-07	3.3e-14	OK
$u_{y'}$ at B	rectangle, Mz = 1	1.5e-07	1.5e-07	2.7e-14	OK
$\theta_{z'}$ at B	rectangle, Mz = 1	1.5e-07	1.5e-07	2.0e-14	OK
$u_{x'}$ at B	circle, Fx = 1	3.183099e-10	3.183099e-10	1.6e-16	OK
$u_{y'}$ at B	circle, Fy = 1	1.706848e-07	1.706848e-07	1.3e-14	OK
$\theta_{z'}$ at B	circle, Fy = 1	1.27324e-07	1.27324e-07	1.2e-14	OK
$u_{z'}$ at B	circle, Fz = 1	1.706848e-07	1.706848e-07	7.3e-15	OK
$\theta_{y'}$ at B	circle, Fz = 1	-1.27324e-07	-1.27324e-07	7.5e-15	OK
$\theta_{x'}$ at B	circle, Mx = 1	1.655211e-07	1.655211e-07	3.2e-16	OK
$u_{z'}$ at B	circle, My = 1	-1.27324e-07	-1.27324e-07	7.3e-15	OK
$\theta_{y'}$ at B	circle, My = 1	1.27324e-07	1.27324e-07	5.4e-15	OK
$u_{y'}$ at B	circle, Mz = 1	1.27324e-07	1.27324e-07	1.2e-14	OK
$\theta_{z'}$ at B	circle, Mz = 1	1.27324e-07	1.27324e-07	8.9e-15	OK

Pass criterion: relative difference ≤ 1e-06 (the formulas are exact for this element).

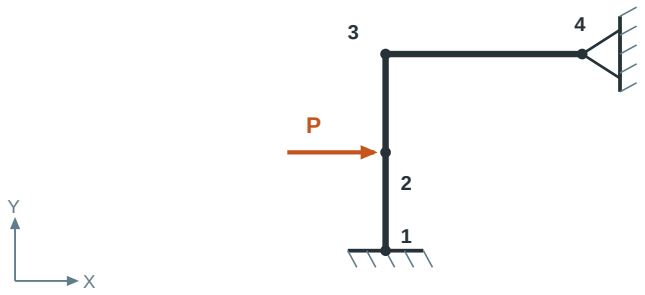
Conclusion

The twenty values are met to the last figure, for both sections. The displacements are given in the axes of the member.

Case 4.6 — RDM, example 1 — Plane portal frame

What this case proves. A column and a beam of the same section, clamped at one end and pinned at the other, under a horizontal force: displacements, rotations and reactions of a frame that is statically indeterminate.

Model



Column of 2 m, beam of 2 m
1 clamped, 4 pinned
P at mid-height of the column

Data	Symbol	Value
Geometry		nodes 1 (0, 0), 2 (0, 1), 3 (0, 2), 4 (2, 2) m
Section	A, I_z	A = 16 cm^2, I_z = 135 cm^4, no shear deformation
Material	E	2.0e11 Pa
Supports		1: Ux = Uy = Uz = Rx = Ry = Rz = 0 4: Ux = Uy = Uz = Rx = Ry = 0
Load	P	10 000 N at the node 2, along +X
Mesh		3 elements, 4 nodes

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 1)

The manual gives the results of RDM – Ossatures for the frame of A. Giet and L. Géminard, Résistance des matériaux, tome 2, 1968.

The displacements along the members are printed with two figures only (0.0017 and 0.0033 mm): they are compared within that rounding.

Results

Quantity	Origin of the reference	Reference	KUSOR	Difference	
Node 2, u [mm]	printed in the manual	2.2144	2.214396	2.0e-06	OK
Node 2, v [mm]	printed in the manual	-0.0017	-0.001666855	1.9e-02	OK
Node 2, θ_z [degrees]	printed in the manual	-0.0388	-0.03880728	1.9e-04	OK
Node 3, u [mm]	printed in the manual	0.0245	0.0245163	6.6e-04	OK
Node 3, v [mm]	printed in the manual	-0.0033	-0.003333711	1.0e-02	OK
Node 3, θ_z [degrees]	printed in the manual	0.151	0.1510151	1.0e-04	OK
Node 4, θ_z [degrees]	printed in the manual	-0.0754	-0.07536427	4.7e-04	OK
Reaction at 1, R_x [N]	printed in the manual	-6077.4	-6077.392	1.3e-06	OK
Reaction at 1, R_y [N]	printed in the manual	533.4	533.394	1.1e-05	OK
Reaction at 1, M_z [N.m]	printed in the manual	3221.6	3221.571	9.0e-06	OK
Reaction at 4, R_x [N]	printed in the manual	-3922.6	-3922.608	2.0e-06	OK
Reaction at 4, R_y [N]	printed in the manual	-533.4	-533.394	1.1e-05	OK

Pass criterion: relative difference ≤ 5e-04, or the rounding of the printed value when it has fewer figures.

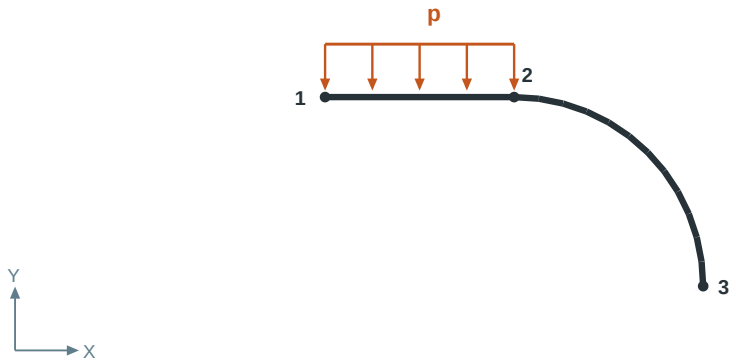
Conclusion

The twelve printed values are met: the displacements and rotations of the three free nodes and the five reactions.

Case 4.7 — RDM, example 3 — Plane ring

What this case proves. A curved member made of straight elements, with shear deformation: the closed-form solution has a bending term, an axial term and a shear term.

Model



A quarter of the ring

1 - 2: straight, L = 50 mm

2 - 3: quarter circle, R = 50 mm

1 on the plane x = 0

3 on the plane y = 0

Data	Symbol	Value
Geometry	L, R	straight part of 50 mm, then a quarter circle of radius 50 mm
Section	c	solid square of 10 mm, shear factor 5/6
Material	E, ν	E = 2.0e11 Pa, ν = 0.3
Supports		1: Ux = Rz = 0 3: Uy = Rz = 0 (planes of symmetry); everywhere Uz = Rx = Ry = 0
Load	p	10 N/mm on the straight part, along –Y
Mesh		1 element on the straight part, 80 straight elements on the quarter circle

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 5)

$$v_1 = (6\pi^2 + 17\pi - 6)\, p\, R^4 / [24\,(2+\pi)\, E\, I_z] + \pi\, p\, R^2 / (4\, E\, A) + (2+\pi)\, p\, R^2 / (4\, G\, A_s)$$

The manual prints M_z = –18 567 N.mm at 3; its own formula, $(2 + 3\pi)\, p\, R^2 / [3\,(2+\pi)]$, gives –18 517 N.mm, which is used.

With 20 elements on the arc, as in the manual, KUSOR gives v_1 = –0.329765 mm and u_3 = 0.133290 mm: the values RDM – Ossatures prints, figure for figure.

Results

Quantity	Origin of the reference	Reference	KUSOR	Difference	
Node 1, v [mm]	formula of the manual	-0.330021	-0.330005	4.8e-05	OK
Node 3, u [mm]	formula of the manual	0.133317	0.1333156	1.1e-05	OK
Reaction at 1, M_z [N.mm]	formula of the manual	-18983	-18983	0.0e+00	OK
Reaction at 3, F_y [N]	formula of the manual	500	500	0.0e+00	OK
Reaction at 3, M_z [N.mm]	formula of the manual	-18517	-18517	0.0e+00	OK
Bending moment in the section 2 [N.mm]	formula of the manual	6483	6483	0.0e+00	OK
Node 1, v without shear areas [mm]	bending and axial terms	-0.325008	-0.324992	4.9e-05	OK

The last line is the same model solved with zero shear areas: the Euler-Bernoulli limit. Pass criterion: relative difference ≤ 5e-04, or the rounding of the printed value when it has fewer figures.

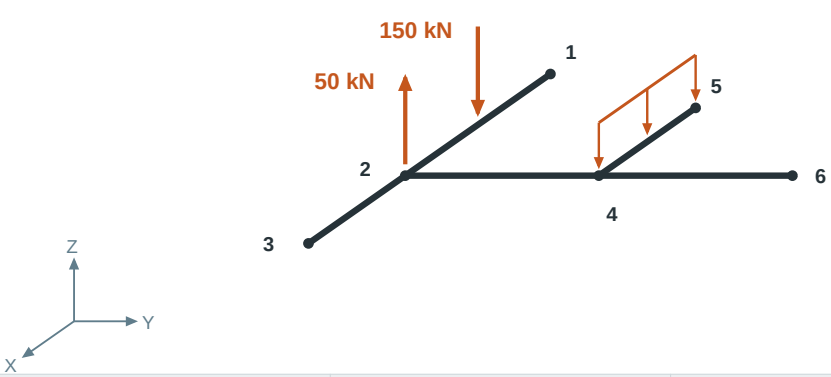
Conclusion

The six values of the closed-form solution are met with 80 elements on the arc; the difference left is that of the straight elements, and it falls as the square of their length. Shear adds 1.5 % to the deflection, and the solution without shear is met as well.

Case 4.8 — RDM, example 4 — Grid of beams

What this case proves. Beams in one plane loaded out of that plane: each beam is bent and twisted, with a force between the nodes of one beam and a load per unit length on another.

Model



A grid in the plane XY

1, 3, 5 and 6 clamped

50 kN and 100 kN.m (about Y) at 2

150 kN along -Z at the middle of 1 - 2

75 kN/m along -Z on 5 - 4

Data	Symbol	Value
Geometry		1 (0, 0), 2 (6, 0), 3 (10, 0), 4 (6, 4), 5 (2, 4), 6 (6, 8) m, in the plane Z = 0
Section	I, J	I = 1e5 cm^4, J = 2e5 cm^4, no shear deformation
Material	E, ν	E = 2.0e11 Pa, ν = 0.25
Supports		1, 3, 5 and 6: Ux = Uy = Uz = Rx = Ry = Rz = 0
Loads		see the drawing
Mesh		5 elements, 6 nodes

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 8)

Results of RDM – Ossatures for a grid of W. Weaver and J. Gere (1990, pages 342-345).

Results

Quantity	Node	Reference	KUSOR	Difference	
w [mm]	node 2	-1.2182	-1.218248	3.9e-05	OK
θ_x [1e-3 rad]	node 2	-0.35599	-0.3559946	1.3e-05	OK
θ_y [1e-3 rad]	node 2	-0.14976	-0.1497583	1.1e-05	OK
w [mm]	node 4	-2.0993	-2.099255	2.2e-05	OK
θ_x [1e-3 rad]	node 4	0.28856	0.2885637	1.3e-05	OK
θ_y [1e-3 rad]	node 4	0.18376	0.1837634	1.9e-05	OK
Reaction F_z [kN]	node 1	93.528	93.52803	3.3e-07	OK
Reaction M_x [kN.m]	node 1	9.493	9.493189	2.0e-05	OK
Reaction M_y [kN.m]	node 1	-163.092	-163.0921	9.0e-07	OK
Reaction F_z [kN]	node 3	34.452	34.45241	1.2e-05	OK
Reaction M_x [kN.m]	node 3	14.24	14.23978	1.5e-05	OK
Reaction M_y [kN.m]	node 3	76.393	76.39274	3.4e-06	OK
Reaction F_z [kN]	node 5	214.94	214.9398	9.8e-07	OK
Reaction M_x [kN.m]	node 5	-11.543	-11.54255	3.9e-05	OK
Reaction M_y [kN.m]	node 5	-239.068	-239.0677	1.0e-06	OK
Reaction F_z [kN]	node 6	57.08	57.07977	4.1e-06	OK
Reaction M_x [kN.m]	node 6	-128.588	-128.5877	2.2e-06	OK
Reaction M_y [kN.m]	node 6	-7.351	-7.350537	6.3e-05	OK

Pass criterion: relative difference ≤ 5e-04, or the rounding of the printed value when it has fewer figures.

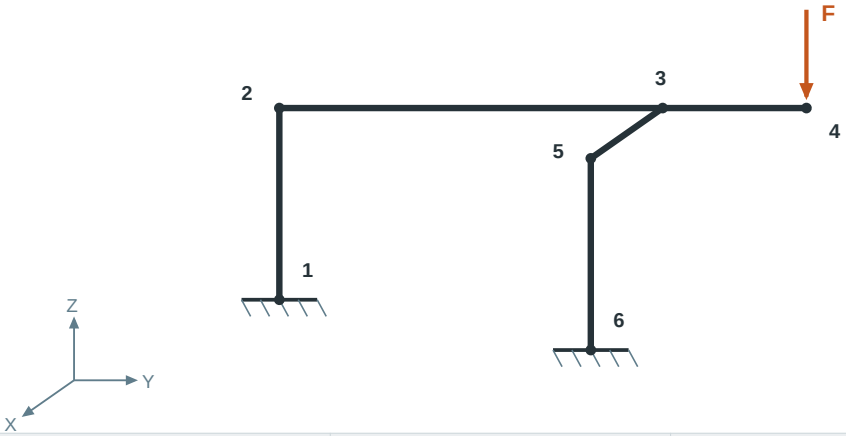
Conclusion

The eighteen printed values are met: deflection and rotations of the two free nodes, and the force and the two moments at each of the four clamped ends.

Case 4.9 — RDM, example 5 — Space frame

What this case proves. A frame in three dimensions, with shear deformation: members along the three axes, three different rectangular sections, bending about both axes and torsion in every member.

Model



Solid rectangles, depth by width

1 - 2: 600 (along Y) by 300 mm

2 - 3 - 4: 600 (along Z) by 300 mm

3 - 5: 500 (along Z) by 300 mm

5 - 6: 800 (along X) by 300 mm

1 and 6 clamped

Data	Symbol	Value
Geometry		1 (0, 0, 0), 2 (0, 0, 4), 3 (0, 8, 4), 4 (0, 11, 4), 5 (3, 8, 4), 6 (3, 8, 0) m
Sections		solid rectangles (see the drawing), shear factor 5/6
Material	E, ν	E = 1.0e11 Pa, ν = 0.2987
Supports		1 and 6: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	F	10 000 N at the node 4, along –Z
Mesh		5 elements, 6 nodes

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 10)

References: the results of RDM – Ossatures, in daN.m, compared in absolute value.

Where it prints a torque that varies along an unloaded member (322.8 then 323.1), the first is used.

Results

Quantity	Member	Reference	KUSOR	Difference	
Torque [daN.m]	1 - 2	5.6	5.59	1.8e-03	OK
M_Y at the origin [daN.m]	1 - 2	271.5	271.4929	2.6e-05	OK
M_Z at the origin [daN.m]	1 - 2	389.7	389.7265	6.8e-05	OK
M_Y at the end [daN.m]	1 - 2	322.8	322.8028	8.7e-06	OK
M_Z at the end [daN.m]	1 - 2	101.2	101.2187	1.8e-04	OK
Torque [daN.m]	2 - 3	322.8	322.8028	8.7e-06	OK
M_Y at the origin [daN.m]	2 - 3	5.6	5.59	1.8e-03	OK
M_Z at the origin [daN.m]	2 - 3	101.2	101.2187	1.8e-04	OK
M_Y at the end [daN.m]	2 - 3	97.04	97.0296	1.1e-04	OK
M_Z at the end [daN.m]	2 - 3	2511	2511.399	1.6e-04	OK
M_Z at the origin [daN.m]	3 - 4	3000	3000	0.0e+00	OK
Torque [daN.m]	3 - 5	488.6	488.6007	1.4e-06	OK
M_Y at the origin [daN.m]	3 - 5	322.8	322.8028	8.7e-06	OK
M_Z at the origin [daN.m]	3 - 5	97.04	97.0296	1.1e-04	OK
M_Y at the end [daN.m]	3 - 5	3581	3581.015	4.2e-06	OK
M_Z at the end [daN.m]	3 - 5	119.4	119.3512	4.1e-04	OK
Torque [daN.m]	5 - 6	119.4	119.3512	4.1e-04	OK
M_Y at the origin [daN.m]	5 - 6	3581	3581.015	4.2e-06	OK
M_Z at the origin [daN.m]	5 - 6	488.6	488.6007	1.4e-06	OK
M_Y at the end [daN.m]	5 - 6	3632	3632.325	8.9e-05	OK
M_Z at the end [daN.m]	5 - 6	200.1	200.0929	3.5e-05	OK

Pass criterion: relative difference ≤ 5e-04, or the rounding of the printed value when it has fewer figures.

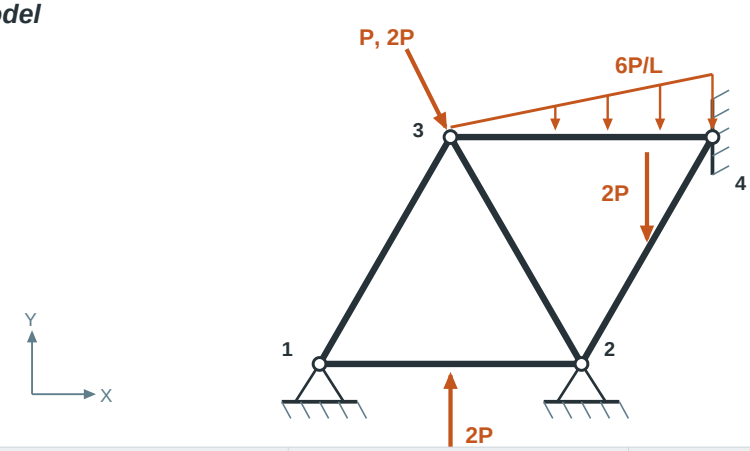
Conclusion

The twenty-one moments are met, most of them to the four figures printed. The values of J.-J. Barrau and S. Laroze (ENSAE, 1984), which the manual also quotes, differ from those of RDM by up to 4 %.

Case 4.10 — RDM, example 7 — Pin-ended members loaded between their nodes

What this case proves. Members hinged at both ends that carry loads between their nodes, a force at mid-length and a load that grows linearly: the hinged members must pass these loads on to the nodes.

Model



Five members of 1.5 m,
hinged at both ends
1 and 2 pinned; at 4, u = 0
P = 1 000 daN
forces 2P at the middle of
1 - 2 and of 2 - 4

Data	Symbol	Value
Geometry	L	two equilateral triangles of side L = 1.5 m
Section	A	hollow square 100 x 5 mm, A = 19 cm^2; no shear deformation
Material	E	2.0e11 Pa
Supports		1, 2: Ux = Uy = 0; 4: Ux = 0; all: Uz = Rx = Ry = Rz = 0
Loads	P	(P, –2P) at 3; (0, 2P) at the middle of 1 - 2; (0, –2P) at the middle of 2 - 4; on 3 - 4, from 0 at 3 to 6P/L at 4, along –Y
Mesh		5 elements released in Rz at both ends, 4 nodes

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 14)

The manual gives the results of RDM – Ossatures for a structure of W. Weaver and J. Gere, Matrix analysis of framed structures, 1990, page 283.

Forces are printed in daN with three or four figures. Shear forces are compared in absolute value, the sign conventions of the two programs being different.

Results

Quantity	Origin of the reference	Reference	KUSOR	Difference	
Node 3, u [mm]	printed in the manual	0.02632	0.02631579	1.6e-04	OK
Node 3, v [mm]	printed in the manual	-0.07895	-0.07894737	3.3e-05	OK
Node 4, v [mm]	printed in the manual	-0.15789	-0.1578947	3.0e-05	OK
Reaction at 1, R_x [daN]	printed in the manual	699	699.3587	5.1e-04	OK
Reaction at 1, R_y [daN]	printed in the manual	211	211.3249	1.5e-03	OK
Reaction at 2, R_x [daN]	printed in the manual	699	699.3587	5.1e-04	OK
Reaction at 2, R_y [daN]	printed in the manual	4789	4788.675	6.8e-05	OK
Reaction at 4, R_x [daN]	printed in the manual	-2398	-2398.718	3.0e-04	OK
N in 3 - 4, at 3 [daN]	printed in the manual	-667	-666.6667	5.0e-04	OK
N in 3 - 4, at 4 [daN]	printed in the manual	-667	-666.6667	5.0e-04	OK
$\lvert T_Y \rvert$ in 3 - 4, at 3 [daN]	printed in the manual	1000	1000	0.0e+00	OK
$\lvert T_Y \rvert$ in 3 - 4, at 4 [daN]	printed in the manual	2000	2000	0.0e+00	OK

Pass criterion: relative difference ≤ 5e-04, or the rounding of the printed value when it has fewer figures.

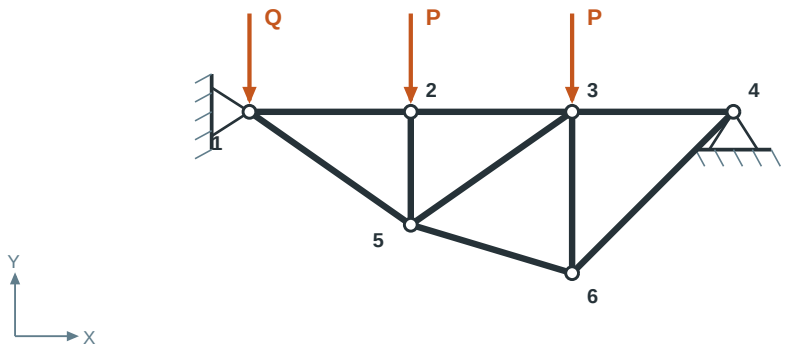
Conclusion

The twelve printed values are met. The displacements are 2/3, 2 and 4 times P L / (E A): the loads on the hinged members reach the nodes as the reactions of simply supported beams.

Case 4.11 — RDM, exercise E1 — Plane truss

What this case proves. A plane truss of nine members hinged at both ends: the reactions and the axial force in every member.

Model



Nine members hinged at both ends

1 pinned; 4 held along Y only

Q = 1 800 daN, P = 3 600 daN

Data	Symbol	Value
Geometry		1 (0, 0), 2 (2, 0), 3 (4, 0), 4 (6, 0), 5 (2, –1.4), 6 (4, –2) m
Section	A	hollow square 70 x 5 mm, A = 13 cm^2
Material	E	2.1e11 Pa
Supports		1: Ux = Uy = 0; 4: Uy = 0; all: Uz = Rx = Ry = Rz = 0
Loads	Q, P	18 000 N at 1 and 36 000 N at 2 and at 3, along –Y
Mesh		9 elements released in Rz at both ends, 6 nodes

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 16)

> Results of RDM – Ossatures for a truss of F. Frey, Analyse des structures et milieux continus, 1985, page 108. Forces are printed in whole daN.

Results

Quantity	Where	Reference	KUSOR	Difference	
Reaction R_y [daN]	node 1	5400	5400	0.0e+00	OK
Reaction R_y [daN]	node 4	3600	3600	0.0e+00	OK
Axial force N [daN]	member 1 - 2	-5143	-5142.857	2.8e-05	OK
Axial force N [daN]	member 2 - 3	-5143	-5142.857	2.8e-05	OK
Axial force N [daN]	member 3 - 4	-3600	-3600	0.0e+00	OK
Axial force N [daN]	member 1 - 5	6278	6277.657	5.5e-05	OK
Axial force N [daN]	member 5 - 6	3758	3758.51	1.4e-04	OK
Axial force N [daN]	member 6 - 4	5091	5091.169	3.3e-05	OK
Axial force N [daN]	member 2 - 5	-3600	-3600	0.0e+00	OK
Axial force N [daN]	member 5 - 3	1883	1883.297	1.6e-04	OK
Axial force N [daN]	member 6 - 3	-4680	-4680	0.0e+00	OK

Pass criterion: relative difference ≤ 5e-04, or the rounding of the printed value when it has fewer figures.

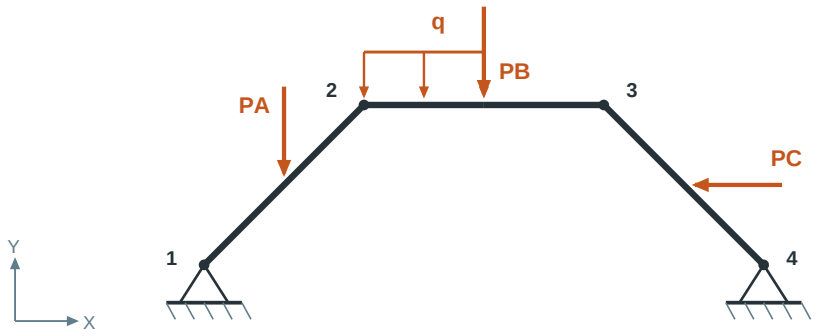
Conclusion

The two reactions and the nine axial forces are met, in tension and in compression, with the sign of the manual: a negative force is a compression.

Case 4.12 — RDM, exercise E2 — Portal with loads on its three members

What this case proves. Forces at mid-length of two sloping members and of a beam, with a load over half of the beam: the reactions, and the largest bending moment, which lies inside the loaded length.

Model



Three HEA 600, welded

1 held along Y only; 4 pinned

PA = PB = 2 000 daN

PC = 1 000 daN

q = 1 000 daN/m on half of 2 - 3

Data	Symbol	Value
Geometry		1 (0, 0), 2 (4, 4), 3 (10, 4), 4 (14, 0) m
Section	A, I	HEA 600: A = 226.5 cm^2, I = 141 200 cm^4; no shear deformation
Material	E	2.1e11 Pa
Supports		1: Uy = 0; 4: Ux = Uy = 0; all: Uz = Rx = Ry = 0
Loads		at mid-length: (0, –20 000) N on 1 - 2 and on 2 - 3, (–10 000, 0) N on 3 - 4; 10 000 N/m along –Y on the first half of 2 - 3
Mesh		3 elements, 4 nodes

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 18)

Results of RDM – Ossatures for a frame of A. Jalil, Calcul pratique des structures, 1985, page 55.

The largest moment is where the shear force is zero: $M = M_2 + V_2^2 / (2\,q)$, from the moment and the shear force KUSOR gives at the end 2 of the beam. That point is at 2.68 m from 2; the manual prints 2.66 m.

Results

Quantity	Origin of the reference	Reference	KUSOR	Difference	
Reaction at 1, R_y [daN]	printed in the manual	4679	4678.571	9.2e-05	OK
Reaction at 4, R_x [daN]	printed in the manual	1000	1000	0.0e+00	OK
Reaction at 4, R_y [daN]	printed in the manual	2321	2321.429	1.8e-04	OK
Largest bending moment, in 2 - 3 [daN.m]	printed in the manual	18301	18301.66	3.6e-05	OK

Pass criterion: relative difference ≤ 5e-04, or the rounding of the printed value when it has fewer figures.

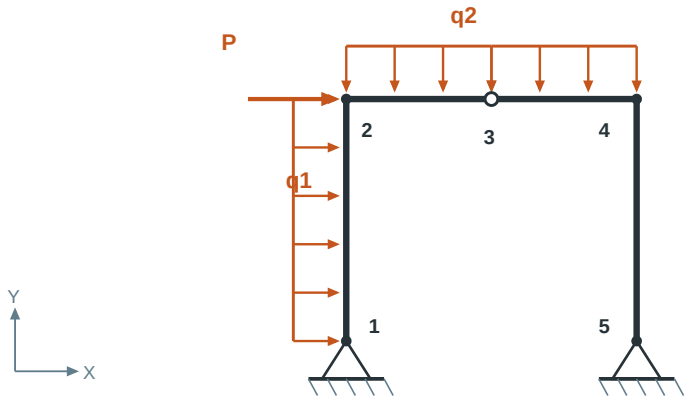
Conclusion

The three reactions and the largest moment are met. The structure is statically determinate, so this case checks the loads on the members and the forces inside them, not the stiffness.

Case 4.13 — RDM, exercise E3 — Portal with three hinges

What this case proves. A portal pinned at its feet with a hinge at the middle of its beam, under a force and loads per unit length on three members: the hinge makes it statically determinate.

Model



Four HEA 600, a hinge at 3

1 and 5 pinned

P = 4 000 daN

q1 = 1 000 daN/m, q2 = 5 000 daN/m

Data	Symbol	Value
Geometry		1 (–3, 0), 2 (–3, 5), 3 (0, 5), 4 (3, 5), 5 (3, 0) m
Section	A, I	HEA 600: A = 226.5 cm^2, I = 141 200 cm^4; no shear deformation
Material	E	2.1e11 Pa
Supports		1 and 5: Ux = Uy = 0; all: Uz = Rx = Ry = 0
Loads	P, q1, q2	40 000 N at 2, along +X; 10 000 N/m on 1 - 2, along +X; 50 000 N/m on 2 - 3 and 3 - 4, along –Y
Mesh		4 elements; the element 2 - 3 is released in Rz at 3

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 19)

Results of RDM – Ossatures for a frame of A. Jalil, Calcul pratique des structures, 1985, page 57.

Results

Quantity	Origin of the reference	Reference	KUSOR	Difference	
Reaction at 1, R_x [daN]	printed in the manual	-1250	-1250	0.0e+00	OK
Reaction at 1, R_y [daN]	printed in the manual	9583	9583.333	3.5e-05	OK
Reaction at 5, R_x [daN]	printed in the manual	-7750	-7750	0.0e+00	OK
Reaction at 5, R_y [daN]	printed in the manual	20417	20416.67	1.6e-05	OK
Bending moment at 4, the largest [daN.m]	printed in the manual	38750	38750	0.0e+00	OK

Pass criterion: relative difference ≤ 5e-04, or the rounding of the printed value when it has fewer figures.

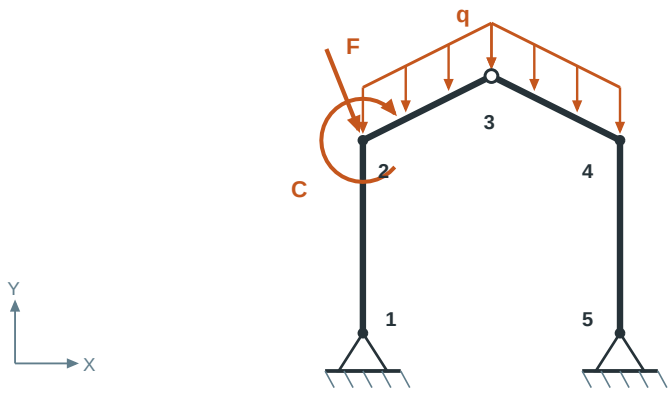
Conclusion

The four reactions and the largest moment are met. With the hinge the result does not depend on the stiffness: this case checks the release and the loads on the members.

Case 4.14 — RDM, exercise E4 — Three-hinged portal with a sloping roof

What this case proves. A load given per metre of span on sloping members, with a force and a moment at a node, on a portal made statically determinate by a hinge at its ridge.

Model



Four HEA 600, a hinge at 3

1 and 5 pinned

F = (2 000, -5 000) daN

C = -3 000 daN.m

q = 1 000 daN per metre of span

Data	Symbol	Value
Geometry		1 (–3, 0), 2 (–3, 4.5), 3 (0, 6), 4 (3, 4.5), 5 (3, 0) m
Section	A, I	HEA 600: A = 226.5 cm^2, I = 141 200 cm^4; no shear deformation
Material	E	2.1e11 Pa
Supports		1 and 5: Ux = Uy = 0; all: Uz = Rx = Ry = 0
Loads	F, C, q	(20 000, –50 000) N and –30 000 N.m at 2; 10 000 N per metre of span on 2 - 3 and 3 - 4, along –Y
Mesh		4 elements; the element 2 - 3 is released in Rz at 3

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 20)

Results of RDM – Ossatures for a frame of A. Jalil, Calcul pratique des structures, 1985, page 6.

KUSOR takes a load per unit length of the member: q is multiplied by 3 / 3.354, the span of a rafter over its length.

Results

Quantity	Origin of the reference	Reference	KUSOR	Difference	
Reaction at 1, R_x [daN]	printed in the manual	-250	-250	0.0e+00	OK
Reaction at 1, R_y [daN]	printed in the manual	6000	6000	0.0e+00	OK
Reaction at 5, R_x [daN]	printed in the manual	-1750	-1750	0.0e+00	OK
Reaction at 5, R_y [daN]	printed in the manual	5000	5000	0.0e+00	OK

Pass criterion: relative difference ≤ 5e-04, or the rounding of the printed value when it has fewer figures.

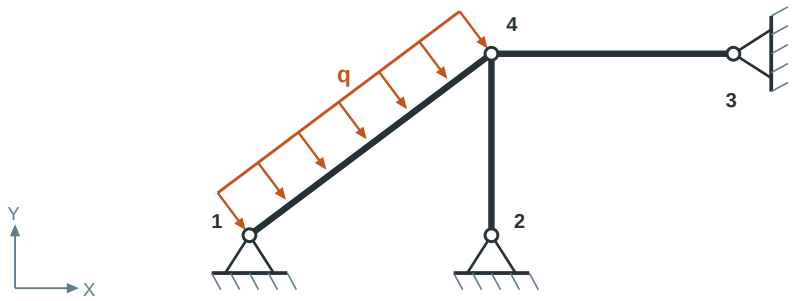
Conclusion

The four reactions are met to the last figure: 3 000 daN on each rafter, whatever its slope.

Case 4.15 — RDM, exercise E8 — Three bars, a load across one of them

What this case proves. A load perpendicular to a hinged member: the member carries it in bending to its two ends and takes no axial force; the two other bars hold the node.

Model



Three members hinged at both ends

1, 2 and 3 pinned

1 - 4: 5 L long, 25 by 20 mm

2 - 4: 3 L long, 15 by 20 mm

4 - 3: 4 L long, 20 by 20 mm

q across 1 - 4

Data	Symbol	Value
Geometry	L	1 (–4L, 0), 2 (0, 0), 3 (4L, 3L), 4 (0, 3L); L = 0.1 m
Sections	a	solid rectangles (see the drawing), a = 10 mm; no shear deformation
Material	E	2.0e11 Pa
Supports		1, 2 and 3: Ux = Uy = 0; all: Uz = Rx = Ry = Rz = 0
Load	q	8 000 N/m on 1 - 4, perpendicular to it, towards the lower right
Mesh		3 elements released in Rz at both ends, 4 nodes

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 25)

$u_4 = 3\,q\,L^2 / (2\,E\,a^2) = 6e\text{-}3$ mm $v_4 = -2\,q\,L^2 / (E\,a^2) = -8e\text{-}3$ mm

$N_{1\text{-}4} = 0$ $N_{2\text{-}4} = -2\,q\,L = -1\,600$ N $N_{4\text{-}3} = -3\,q\,L / 2 = -1\,200$ N

Results

Quantity	Formula	Reference	KUSOR	Difference	
Node 4, u [mm]	$3\,q\,L^2 / (2\,E\,a^2)$	0.006	0.006	1.4e-16	OK
Node 4, v [mm]	$-2\,q\,L^2 / (E\,a^2)$	-0.008	-0.008	2.2e-16	OK
Axial force in 1 - 4 [N]	0	0	0	0.0e+00	OK
Axial force in 2 - 4 [N]	–2 q L	-1600	-1600	0.0e+00	OK
Axial force in 4 - 3 [N]	–3 q L / 2	-1200	-1200	0.0e+00	OK

Pass criterion: relative difference ≤ 1e-06 (the formulas are exact).

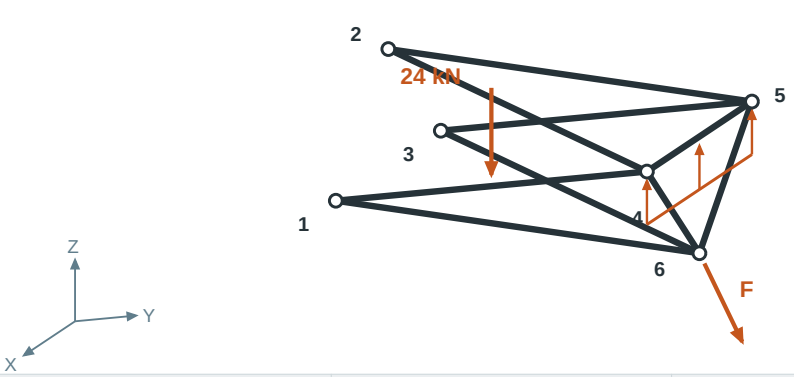
Conclusion

The five values of the closed-form solution are met: the loaded member brings half of its load to the node 4, as a force perpendicular to it, and stays free of axial force.

Case 4.16 — RDM, exercise E10 — Space truss with loaded members

What this case proves. A truss in three dimensions whose members are hinged at both ends, two of them carrying a load between their nodes: a force at mid-length and a load per unit length.

Model



Nine members hinged at both ends

1, 2 and 3 held on a wall, Y = 0

F = (48, 24, -24) kN at 6

24 kN along -Z at the middle of 1 - 4

24 kN/m along +Z on 4 - 5

Data	Symbol	Value
Geometry		1 (3, 0, 0), 2 (0, 0, 3), 3 (–3, 0, 0), 4 (3, 5, 0), 5 (–3, 5, 0), 6 (0, 5, –3) m
Sections	A	4 - 5, 4 - 6, 5 - 6: 100 cm^2; the others: 200 cm^2
Material	E, ν	E = 8.0e10 Pa, ν = 0.3
Supports		1, 2 and 3: Ux = Uy = Uz = 0 (ball joints)
Loads		see the drawing
Mesh		9 elements released in Ry and Rz at both ends; node rotations held

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 27)

Results of RDM – Ossatures for a truss of W. Weaver and J. Gere (1990, page 352).

A ball joint also frees the rotation about the axis of the member, which no load calls on here: it is kept, and the rotations of the nodes are held, so that no member can spin.

Results

Quantity	Node	Reference	KUSOR	Difference	
Displacement u_x [mm]	node 4	0.44359	0.4435858	9.4e-06	OK
Displacement u_y [mm]	node 4	0.30312	0.303125	1.6e-05	OK
Displacement u_z [mm]	node 4	2.08842	2.088422	1.1e-06	OK
Displacement u_x [mm]	node 5	0.02059	0.02058581	2.0e-04	OK
Displacement u_y [mm]	node 5	0.33437	0.334375	1.5e-05	OK
Displacement u_z [mm]	node 5	1.79382	1.793821	8.2e-07	OK
Displacement u_x [mm]	node 6	0.41121	0.411206	9.6e-06	OK
Displacement u_y [mm]	node 6	1.34562	1.345617	2.2e-06	OK
Displacement u_z [mm]	node 6	2.10171	2.10171	9.6e-08	OK
Reaction R_x [kN]	node 1	-13.8	-13.8	0.0e+00	OK
Reaction R_y [kN]	node 1	-74	-74	0.0e+00	OK
Reaction R_z [kN]	node 1	-1.8	-1.8	0.0e+00	OK
Reaction R_x [kN]	node 2	-6	-6	0.0e+00	OK
Reaction R_y [kN]	node 2	204	204	0.0e+00	OK
Reaction R_z [kN]	node 2	-122.4	-122.4	0.0e+00	OK
Reaction R_x [kN]	node 3	-28.2	-28.2	0.0e+00	OK
Reaction R_y [kN]	node 3	-154	-154	0.0e+00	OK
Reaction R_z [kN]	node 3	28.2	28.2	0.0e+00	OK

Pass criterion: relative difference ≤ 5e-04, or the rounding of the printed value when it has fewer figures.

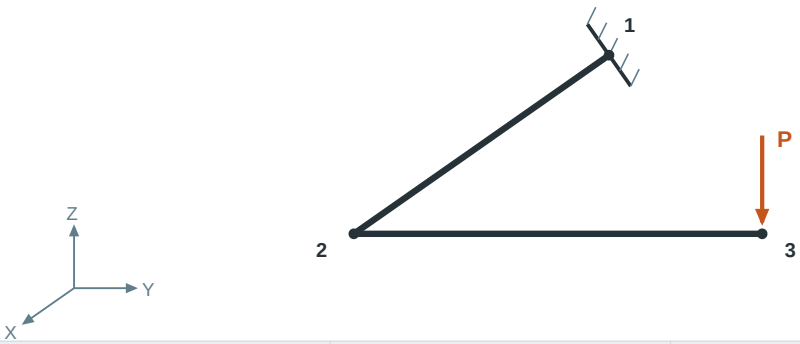
Conclusion

The eighteen printed values are met, the nine reactions to the last figure: the loads on the members 1 - 4 and 4 - 5 are shared between their nodes.

Case 4.17 — RDM, exercise S11 — Bending, torsion and shear together

What this case proves. A force out of the plane of two members at a right angle: one member is bent and twisted, the other is bent, and the closed-form deflection has a bending term, a torsion term and a shear term.

Model



Two members at a right angle

1 - 2 along X, L = 0.5 m

2 - 3 along Y, H = 0.4 m

1 clamped

P at 3, along -Z

Data	Symbol	Value
Geometry	L, H	1 (0, 0, 0), 2 (L, 0, 0), 3 (L, H, 0); L = 0.5 m, H = 0.4 m
Section	c	solid square of 40 mm: A = c^2, I = c^4/12, J = 0.140577 c^4, shear factor 5/6
Material	E, ν	E = 2.0e11 Pa, ν = 0.3
Supports		1: Ux = Uy = Uz = Rx = Ry = Rz = 0
Load	P	3 000 N at the node 3, along –Z
Mesh		2 elements, 3 nodes

Reference solution (RDM – Ossatures, exercise manual (Y. Debard, IUT du Mans, 2011), page 46)

$$w_2 = P L^3 / (3 E I) + P L / (G A_s)$$

$$w_3 = P (L^3 + H^3) / (3 E I) + P H^2 L / (G J) + P (L + H) / (G A_s)$$

Results

Quantity	Model	Reference	KUSOR	Difference	
Node 2, w [mm]	with shear deformation	-2.94431	-2.944312	8.5e-07	OK
Node 3, w [mm]	with shear deformation	-13.12564	-13.12564	1.2e-07	OK
Node 2, w [mm]	without	-2.92969	-2.929688	8.5e-07	OK
Node 3, w [mm]	without	-13.09931	-13.09931	2.6e-07	OK

Pass criterion: relative difference ≤ 5e-04, or the rounding of the printed value when it has fewer figures.

Conclusion

The four values of the manual are met to the seven figures printed, with and without shear deformation. The twist of the member 1 - 2 gives more than half of the deflection of the node 3.